Predicting bearing states in three dimensions

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Bichromatic bearings have an infinite number of sliding-free states, so called bearing states. For three-dimensional bichromatic bearings whose bearing states have four degrees of freedom, we show how the bearing state can be analytically predicted from the initial state without any information about the nature of the contact forces. We provide a systematic way of constructing such bearings and also show how the bearing state is modified by blocking a single sphere and that any bearing state can be induced by controlling the angular velocities of only two spheres. Furthermore, we show that it is possible to determine the total mass and the center of mass of the bearing by analyzing its response to changes of the angular velocities of at most two spheres.

Bearings are machine elements that allow different parts to support (to bear) each other to constrain motion and reduce friction. Most common are plain bearings [1], fluid bearings [2], magnetic bearings [3], and ball bearings [4]. In the latter, spheres are placed between moving parts to prevent sliding friction between them but allow rolling on the spheres instead, where the individual spheres are typically isolated from each other. There also exist general models of two- and three-dimensional bearings consisting of many disks [5,7] or spheres [8,9] which are in direct contact. They have been used to explain the existence of seismic gaps [10], zones between tectonic plates which show unexpected low seismic activity [11,12]. To minimize friction, bearings have to relax toward a rolling state called bearing state, where pairs of touching spheres have the same tangential velocity at their contact point and thus vanishing sliding friction. A spatial arrangement of spheres is a bearing solely if it has bearing states. In a bichromatic packing of spheres, bearing states always exist [8]. Bichromatic means one can color the spheres using only two colors, e.g. blue and white, such that there is no contact between spheres of same color. As we will show in the following, bichromatic bearings exhibit four time invariant scalar quantities, allowing to predict the final bearing state for spatial arrangements which we call four-degrees-of-freedom (4DOF) bearings.

We assume spheres are perfectly rigid and although spheres can rotate around any axis, the positions of their centers are fixed at all times. Between touching spheres we only consider sliding friction, first neglecting both rolling and torsion friction for simplicity. We will later comment about torsion friction.

Let us first consider two single spheres in contact as shown in Fig. 1. As long as the bearing state is not reached yet, i.e., the tangential velocities of the two spheres at their contact point are different, sliding forces at the contact point tend to reduce this velocity difference. As follows from Newton’s third law of motion, the sliding force which acts on one sphere is exactly opposite to the one acting on the other sphere. For each sphere the corresponding sliding force will produce a torque \( \vec{T} = \vec{r} \times \vec{F} \), where \( \vec{F} \) points from the center of the sphere to the contact point. The torques acting on the two spheres are parallel to each other and their magnitude is proportional to the radius of the corresponding sphere. Using the law of motion \( \vec{T} = I \vec{a} \) we find

\[
\frac{I_1 \vec{a}_1}{r_1} = \frac{I_2 \vec{a}_2}{r_2},
\]

where \( I_1, I_2, \vec{a}_1, \vec{a}_2, r_1, \) and \( r_2 \) are the moments of inertia, the angular accelerations, and the radii of the first and the second sphere, respectively. Equation (1) holds independently of the sliding forces. Note that due to the constraint that the centers of the spheres are fixed at all times, the sliding forces lead to torques on the two spheres.
spheres, that do not cancel each other, such that the total angular momentum of the system is not conserved.

Let us consider bichromatic bearings that consist of arbitrary many spheres. A sphere might have multiple contacts and therefore multiple simultaneously acting sliding forces. Each sliding force will result in an individual torque and therefore in an individual contribution to the total angular acceleration of the sphere. We define $\alpha^k_i$ as the contribution to the total angular acceleration $\alpha_i$ due to the sliding force at contact $k$, such that $\alpha_i = \sum_k \alpha^k_i$, where the sum runs over all contacts. We can write Eq. (1) analogously for a contact $k$ between spheres $i$ and $j$ with the contributions $\alpha^k_i$ and $\alpha^k_j$. For simplicity we will in the following consider that the moment of inertia of a sphere $i$ is proportional to its mass $m_i$ and its radius squared, as true for e.g. spheres of homogeneous density or hollow spheres. We find

$$m_ir_i\alpha^k_i - m_jr_j\alpha^k_j = \vec{0}. \quad (2)$$

Summing this equation over all contacts $k$, we obtain

$$\sum_i s_im_ir_i\alpha_i = \vec{0}, \quad (3)$$

where the sum runs over all spheres and $s_i$ is $+1$ if sphere $i$ is blue and $-1$ if it is white. Using Eq. (3) we define

$$\vec{B} := \sum_i s_im_ir_i\vec{\omega}_i, \quad (4)$$

where $\vec{\omega}_i$ is the angular velocity of the sphere $i$. Equation (3) shows that $\partial\vec{B}/\partial t = \vec{0}$, i.e. $\vec{B}$ is time invariant.

Let us derive a further time invariant quantity. First, we introduce the position vector $\vec{x}_i$ of a sphere $i$. We choose the center of mass $\vec{M}$ of the entire bearing to be the origin of our coordinate system such that $\vec{x}_i$ points from the center of mass $\vec{M} = \vec{0}$ to the center of sphere $i$. We now multiply (dot product) Eq. (2) by $\vec{x}_i$ and obtain

$$m_ir_i\alpha^k_i \cdot \vec{x}_i - m_jr_j\alpha^k_j \cdot \vec{x}_i = \vec{0}. \quad (5)$$

Since the vector $\vec{x}_{ij} = \vec{x}_j - \vec{x}_i$ is perpendicular to $\alpha^k_i \cdot \vec{x}_i$ and $\alpha^k_j \cdot \vec{x}_j$, it vanishes. If we subtract it from the previous equation, we get

$$m_ir_i\alpha^k_i \cdot \vec{x}_i - m_jr_j\alpha^k_j \cdot \vec{x}_j = \vec{0}. \quad (6)$$

Analogous to the derivation of $\vec{B}$ we use this equation to define a time invariant scalar quantity $H$ as

$$H := \sum_i s_im_ir_i\vec{\omega}_i \cdot \vec{x}_i. \quad (7)$$

We now identify spatial arrangements that are 4DOF bearings, i.e., their bearing state is restricted to four degrees of freedom, such that one can predict the final bearing state using $\vec{B} \in \mathbb{R}^3$ and $H \in \mathbb{R}$ which together represent four time invariant scalar quantities. The only condition for the bearing state of two spheres $i$ and $j$ is that their tangential velocities at their contact point have to be equal. We can formulate this as

$$r_i\dot{x}_{ij} \times \vec{\omega}^b_{ij} = r_j\dot{x}_{ji} \times \vec{\omega}^b_{ji}, \quad (8)$$

where $r_{ij}$ is the unit vector pointing from sphere $i$ to sphere $j$, and $\vec{\omega}^b_{ij}$ is the angular velocity of sphere $i$ in the bearing state (bs). We rewrite this condition in analogy to Ref. [3] as

$$s_ir_i\vec{\omega}^b_{ij} - s_jr_j\vec{\omega}^b_{ji} = c_{ij}\vec{x}_{ij}, \quad (9)$$

where $c_{ij} \in \mathbb{R}$ uniquely relates $\vec{\omega}^b_{ij}$ to $\vec{\omega}^b_{ji}$. One can follow that for every bichromatic bearing, the bearing state is uniquely defined if one knows the angular velocity $\vec{\omega}^b_{ij}$ of a single sphere $i$ and all the parameters $c_{ij}$ that uniquely relate the angular velocities of two touching spheres $i$ and $j$. Since $\vec{\omega}^b_{ij} \in \mathbb{R}^3$, the number of degrees of freedom of the bearing state is equal to three plus the number of degrees of freedom of the set of $c_{ij}$’s of the contacts. Therefore, a bichromatic bearing is a 4DOF bearing only if the set of $c_{ij}$’s is restricted to one degree of freedom. For an open chain of spheres, we have an independent parameter $c_{ij}$ for each contact as shown in Fig. 2, such that every contact adds an additional degree of freedom, not resulting in a 4DOF bearing. But for a bichromatic loop, that contains an even number of spheres $N \geq 4$, the $c_{ij}$’s of the contacts are not independent anymore. Since for every contact Eq. (9) needs to hold, one finds the constraint

$$c_{12}\vec{x}_{12} + c_{23}\vec{x}_{23} + \ldots + c_{N1}\vec{x}_{N1} = \vec{0}, \quad (10)$$

where the indexes $1$ to $N$ are given to the spheres in consecutive order around the loop. The set of $c_{ij}$’s has $N - R$ degrees of freedom, where $R$ is the rank of the $3 \times N$ matrix $(\vec{x}_{12} \vec{x}_{23} \ldots \vec{x}_{N1})$. $R$ is equal to the number of linear independent columns of the matrix and is either two, in case all the centers of the spheres are coplanar, or three otherwise. Given $N \geq 4$ and $R \leq 3$, the only loop that has $N - R = 1$ and therefore is a 4DOF bearing, is a loop of size $N = 4$ where the centers of the spheres are not coplanar ($R = 3$) as the one shown in Fig. 2, which we denote as non-coplanar-4 loop. In that case the $c_{ij}$’s are enforced to be equal to one single parameter $c$. Starting from the pair of spheres as the simplest 4DOF bearing, every other 4DOF bearing including the non-coplanar-4
loop can be iteratively constructed given the following
two minimal conditions of connecting assuming the final
bearing is dichromatic. Either connect two 4DOF bear-
ings by establishing two contacts involving two spheres
of each bearing such that the centers of the four spheres
involved are not coplanar or connect a single sphere via
two contacts to a 4DOF bearing. These ways of connect-
ing ensure that the $c_{ij}$’s of all contacts are enforced to be
equal to a single $c$. The bearing states of 4DOF bearings
can be uniquely described by
\begin{equation}
\omega^\text{bs}_i = \frac{s_i}{r_i} \left( \vec{\Omega} + c\vec{x}_i \right),
\end{equation}
where $\vec{\Omega}$ is a vectorial reference quantity. Details to the
construction of 4DOF bearings and the derivation of Eq.
\ref{eq:8} are shown in the appendix A and B.

Let us now express the bearing state of 4DOF bearings
as a function of $\vec{B}$ and $H$. We first multiply Eq. \ref{eq:8}
by $s_i m_i r_i$ and sum over all spheres to end up with $\vec{B}$
on the left hand side and simultaneously eliminate $c$ from
the right hand side, since $\sum_i c m_i \vec{x}_i = c\vec{M} = \vec{0}$. We then
find $\vec{\Omega} = \vec{B}/M$, where $M = \sum_i m_i$ is the total mass of
all spheres. Second, we multiply (dot product) Eq. \ref{eq:8}
by $s_i m_i r_i \vec{x}_i$ and sum over all spheres to end up with $H$
on the left hand side and eliminate $\vec{\Omega}$ on the right hand
side to find $\vec{c} = H/I$, where $I = \sum_i m_i |\vec{x}_i|^2$. We can now
express the angular velocities of the bearing state as
\begin{equation}
\omega^\text{bs}_i = \frac{s_i}{r_i} \left( \vec{B} M + H I \vec{x}_i \right).
\end{equation}
Since everything on the right hand side of Eq. \ref{eq:9} is time
invariant, we have found an analytic expression for the
final bearing state using $\vec{B}$ and $H$ defined in Eqs. \ref{eq:4}
and \ref{eq:5}. The bearing state depends only on the angular velocities,
the masses of the spheres, and the geometry of the bearing,
but is independent of the type of sliding forces.
Nevertheless, different sliding forces lead to different ki-
etic pathways toward the bearing state as illustrated in
Fig. 3. We show how a non-coplanar-4 loop, a simple
4DOF bearing, relaxes from a random initial configura-
tion toward the bearing state predicted by Eq. \ref{eq:9}
for two different types of sliding forces. In any case, the sliding
force $\vec{F}_{ij}$ that acts on sphere $j$ due to contact with sphere
$i$ always points in the direction opposite to $\vec{v}_{ij} = \vec{v}_j - \vec{v}_i$.
We considered the simple case of a constant force and
one which is proportional to the relative velocity at the contact.
For the latter, we considered $\vec{F}_{ij} = -\sigma \vec{v}_{ij}$ with
$\sigma = 3$ and for the former $\vec{F}_{ij} = -\sigma \vec{v}_{ij}$ with $\sigma = 0.05$, and
$\vec{v}_{ij}$ being the unit vector along $\vec{v}_{ij}$.

Figure 4 pictures the role of $\vec{B}$ and $H$ regarding the
bearing state predicted by Eq. \ref{eq:9}. For $H = 0$ all angular velocities
are parallel to $\vec{B}$, whereas for $H \neq 0$ they depend linearly
on the positions of the spheres such that the rotation axes all meet at
the position $\vec{x}_i = I\vec{B}/(HM)$, at which a sphere $i$ would be at rest, i.e., $\omega^\text{bs}_i = \vec{0}$.
the previous (old) values when blocking sphere \( i \) (details in the appendix D) as

\[
H_{\text{new}} = \frac{H_{\text{old}} - \tilde{B}_{\text{old}} \cdot \tilde{x}_i}{1 + M|\tilde{x}_i|^2/I}
\]  

(10)

and

\[
\tilde{B}_{\text{new}} = -MH_{\text{new}}\tilde{x}_i/I.
\]  

(11)

Furthermore, knowing how to predict the bearing state allows us to impose any desired bearing state. To reach any possible bearing state of a 4DOF bearing, we only need to control the angular velocities of two arbitrarily chosen spheres by either permanently fixing the two angular velocities according to a desired bearing state from Eq. (9) or applying specific changes to them as described in the following. By controlling the angular velocity \( \tilde{\omega}_1 \in \mathbb{R}^3 \) of a single sphere one can already impose bearing states within a three-dimensional subset that can be directly derived from Eqs. (4) and (5). To derive what changes \( \Delta \tilde{\omega}_1 \) and \( \Delta \tilde{\omega}_j \) in angular velocity of any two spheres \( i \) and \( j \) one needs to apply to induce a desired change \( \Delta \tilde{B} \) and \( \Delta H \), one can first employ Eqs. (4) and (5) to calculate the corresponding changes \( \Delta \tilde{B} \) and \( \Delta H \) imposed by a change \( \Delta \tilde{\omega}_j \) in the angular velocity of the sphere \( i \). The individually imposed changes in \( \tilde{B} \) and \( H \) need to sum up to the desired change, i.e., 

\[
\Delta \tilde{B} = \Delta \tilde{B} + \Delta \tilde{B} \quad \text{and} \quad \Delta H = \Delta H + \Delta H.
\]

One can choose an arbitrary unit vector \( \Delta \tilde{\omega}_i \) to define the axis along which one will change the angular velocity of sphere \( i \) given that \( \Delta \tilde{\omega}_i \cdot \tilde{x}_{ij} \neq 0 \). The changes \( \Delta \tilde{\omega}_i \) and \( \Delta \tilde{\omega}_j \) which one needs to apply to impose a desired change \( \Delta \tilde{B} \) and \( \Delta H \) can then be obtained (details in the appendix D) as

\[
\begin{align*}
\Delta \tilde{\omega}_i &= \frac{\Delta \tilde{B} \cdot \tilde{x}_{ij} - \Delta H}{s_i m_i r_i \Delta \tilde{\omega}_j} \Delta \tilde{\omega}_j, \\
\Delta \tilde{\omega}_j &= \frac{\Delta \tilde{B} - s_i m_i r_i \Delta \tilde{\omega}_j}{(s_j m_j r_j)}.
\end{align*}
\]  

(12)

where \( \Delta \tilde{\omega}_j \) is the unit vector along the change \( \Delta \tilde{\omega}_i \), which can be chosen arbitrarily as long as \( \Delta \tilde{\omega}_i \cdot \tilde{x}_{ij} \neq 0 \).

Finally, we show how to determine the total mass of the bearing and its center of mass by having access to not more than two spheres. One could imagine a bearing of arbitrary size, to which one has access through some outer spheres. Using Eqs. (4), (5) and (9), one can determine what will be the change \( \Delta \tilde{\omega}_i^{\text{bs}} \) of the angular velocity of sphere \( i \) from the initial to the next bearing state that results if one applies a change \( \Delta \tilde{\omega}_i \) to the angular velocity of sphere \( i \). Then, starting from any bearing state, one can separately apply two changes \( \Delta \tilde{\omega}_1 \) (first) and \( \Delta \tilde{\omega}_2 \) (second) to an accessible sphere \( i \), wait after each change till the bearing state is reached, and determine the corresponding changes \( \Delta \tilde{\omega}_1^{\text{bs}} \) and \( \Delta \tilde{\omega}_2^{\text{bs}} \) in angular velocity from the previous to the new bearing state. One can then determine the total mass of the bearing \( M \) (details in the appendix E) as

\[
M = (B + \sqrt{B^2 - 4AC})/(2A),
\]

\[
A = |a_2 b_1 - a_1 b_2|, \quad B = |a_2 - a_1|, \quad C = |b_2 - b_1|,
\]

\[
a_n = |\Delta_n \tilde{\omega}_i|/m_i^2 \Delta_n \omega_i, \quad b_n = \Delta_n \omega_i \Delta_n \omega_i/|\Delta_n \omega_i|.
\]  

(13)

We furthermore want to locate the center of mass of the bearing. Having determined the total mass \( M \) we can find a vector \( \tilde{x}_i \) that is parallel to the vector \( \tilde{x}_i \) pointing from the center of mass to the center of sphere \( i \) using Eqs. (4), (5) and (9), where

\[
\tilde{x}_i = \Delta \tilde{\omega}_i^{\text{bs}} - m_i \Delta \tilde{\omega}_i/M.
\]  

(14)

By applying an additional change \( \Delta \tilde{\omega}_j \) to a second accessible sphere \( j \), one can use Eq. (14) to obtain a vector \( \tilde{x}_j \) parallel to \( \tilde{x}_i \). The center of mass is the only point in space that can be reached both going along \( \tilde{x}_i \) from the center of sphere \( i \) and going along \( \tilde{x}_j \) from the center of sphere \( j \), in the general case where the centers of the two spheres and the center of mass of the bearing are not aligned.

So far, we neglected torsion friction, which if considered would exist in bearing states since for two touching spheres \( i \) and \( j \) the torsion torque \( \tau_{ij} \) acting on sphere \( i \) is proportional to \( \left( \tilde{\omega}_i - \tilde{\omega}_j \right) \cdot \tilde{x}_{ij} \). The center of mass is the only point in a general non-zero. Using Eq. (9) one can write the torsion torque for any contact as a function of \( \tilde{B} \) and \( \tilde{H} \), and finds that 

\[
\tau_{ij} = \sum_a (\tilde{\omega}_a - \tilde{\omega}_j) \cdot \tilde{x}_{ij}.
\]

(13)

where we use \( \tau = (\tau_{ij} + |\tau_{ij}|) \), the sum running over all contacts.

In summary, we have defined four time invariant scalar quantities present in bichromatic bearings and have shown how one can use them to predict the final bearing state for any spatial arrangement whose bearing state has four degrees of freedom. Surprisingly, in these bearings the angular velocities of the final bearing state do not depend on the nature of the sliding force, i.e., no matter how the system relaxes toward the bearing state, the final state does only depend on the initial angular velocities. We showed that if one permanently blocks a single sphere, the bearing will relax to a bearing state in which the rotation axes of all spheres go through the center of the blocked sphere. Knowing how to predict the final state allows us to impose any desired configuration by applying changes in angular velocities to only two spheres. Finally we explained that one can, starting from any bearing state, determine the total mass (with access to only one sphere) and the center of mass of the bearing (with access to an additional sphere) by applying random changes to the rotation of the spheres and monitoring the evolution toward the bearing state. Future work might explore possible applications that make use of the controllability of bearings whose bearing state is restricted to four degrees of freedom.
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Appendix A: Characterization of bearing state with all $c_{ij} = c$

The angular velocities $\omega_{ij}^{bs}$ and $\omega_{ji}^{bs}$ of two contacting spheres $i$ and $j$ in the bearing state (bs) are uniquely related to each other via a parameter $c_{ij}$ according to

$$s_j r_j \omega_{ij}^{bs} - s_i r_i \omega_{ji}^{bs} = c_{ij} \vec{x}_{ij},$$

where $r_i$ is the radius of the sphere $i$, $s_i$ is +1 if sphere $i$ is blue and −1 if it is white, and $\vec{x}_{ij} = \vec{x}_j - \vec{x}_i$ is the vector pointing from the center $\vec{x}_i$ of sphere $i$ to the center $\vec{x}_j$ of sphere $j$. If all parameters in the bearing $c_{ij} = c$, one finds for contacting spheres $i$ and $j$

$$s_j r_j \omega_{ij}^{bs} - s_i r_i \omega_{ji}^{bs} = c \vec{x}_{ij},$$

Let us relate the angular velocities of a sphere $i$ and $k$ which are not in contact, but are both contacting a sphere $j$. We sum Eq. (A2) and an analogous equation for the contact between sphere $j$ and $k$ to find

$$s_j r_j \omega_{ij}^{bs} - s_i r_i \omega_{ji}^{bs} + s_k r_k \omega_{jk}^{bs} - s_j r_j \omega_{jk}^{bs} = c \vec{x}_{ij} + c \vec{x}_{jk},$$

$$s_k r_k \omega_{jk}^{bs} - s_i r_i \omega_{ij}^{bs} = c \vec{x}_{ik},$$

which shows the same relation as Eq. (A2) for the non-contacting spheres $i$ and $k$. Therefore, Eq. (A2) is valid for any pair of spheres that is part of the bearing. We can now express the angular velocity of sphere $j$ as a function of the angular velocity of any other sphere $i$ as

$$\omega_{ij}^{bs} = \frac{s_j}{r_j} \left( s_i r_i \omega_{ji}^{bs} + c \vec{x}_{ij} \right).$$

To describe the bearing state in a general way, we can instead of relating to a sphere $i$ of the bearing replace $s_i r_i \omega_{ji}^{bs}$ with a vectorial reference quantity $\vec{\Omega}$ and $\vec{x}_{ij}$ with $\vec{x}_j$ to obtain

$$\omega_{ij}^{bs} = \frac{s_j}{r_j} \left( \vec{\Omega} + c \vec{x}_j \right).$$

Appendix B: Construction of 4DOF bearings

In the following two sections we explain a way how any 4DOF bearing can be constructed and give some practical algorithmic suggestions to generate these bearings numerically.

1. One way to construct any 4DOF bearing

In a 4DOF bearing, Eq. (A2) holds for any two spheres $i$ and $j$ as explained in the former section. To build a more complex 4DOF bearing, one can connect two given 4DOF bearings following a certain sufficient connecting rule. Before two 4DOF bearings $A$ and $B$ are connected, each of them has an independent parameter $c$ which we call $c_A$ and $c_B$, that describe the bearing state according to Eq. (A2), which holds for any two spheres $i$ and $j$ in the bearing as we followed in the previous section. If we want to form another 4DOF bearing by connecting $A$ and $B$, we need to make sure that the way of connecting enforces $c_A = c_B$ for the bearing state. This can be done by involving two spheres of each bearing to couple the two parameters $c_A$ and $c_B$. To involve two spheres of each bearing we need to establish at least two contacts. If we connect sphere $i$ and $j$ from bearing $A$ to sphere $k$ and $l$ from bearing $B$ as shown on the left hand side of Fig. 5, and consider the fact that $A$ and $B$ are 4DOF bearings with bearing parameters $c_A$ and $c_B$, respectively, we can establish the following constraints,

$$s_j r_j \omega_{ij}^{bs} - s_i r_i \omega_{ji}^{bs} = c_A \vec{x}_{ij},$$

$$s_i r_i \omega_{ij}^{bs} - s_j r_j \omega_{ji}^{bs} = c_B \vec{x}_{ij},$$

$$s_k r_k \omega_{jk}^{bs} - s_j r_j \omega_{jk}^{bs} = c_B \vec{x}_{lk},$$

$$s_j r_j \omega_{ik}^{bs} - s_k r_k \omega_{ik}^{bs} = c_k \vec{x}_{ki},$$

which if combined lead to the constraint

$$c_A \vec{x}_{ij} + c_B \vec{x}_{ik} + c_B \vec{x}_{lk} + c_k \vec{x}_{ki} = \vec{0}. \tag{B5}$$

Only if one finds three linear independent vectors among $\vec{x}_{ij}$, $\vec{x}_{jl}$, $\vec{x}_{lk}$, and $\vec{x}_{ki}$, i.e., if the centers of the four spheres $i$, $j$, $k$, and $l$ are not coplanar, Eq. (B5) enforces $c_A = c_B = c$ for the bearing state. Note that spheres $i$ and $j$ of the bichromatic bearing $A$ can be of any color, as long as the final bearing is bichromatic. Another way to extend a 4DOF bearing is to integrate a single sphere $k$ by establishing contacts to two spheres $i$ and $j$ of a bearing $A$, as shown on the right of Fig. 5. The constraints

$$s_j r_j \omega_{ij}^{bs} - s_i r_i \omega_{ji}^{bs} = c_A \vec{x}_{ij},$$

$$s_k r_k \omega_{jk}^{bs} - s_j r_j \omega_{jk}^{bs} = c_j \vec{x}_{jk},$$

$$s_i r_i \omega_{ij}^{bs} - s_k r_k \omega_{ik}^{bs} = c_k \vec{x}_{ki},$$

$$s_j r_j \omega_{ij}^{bs} - s_k r_k \omega_{ik}^{bs} = c_{ik} \vec{x}_{ki}.$$
can be added to obtain
\[ c_A \vec{x}_{ij} + c_{jk} \vec{x}_{jk} + c_{ik} \vec{x}_{ki} = \vec{0}, \tag{B9} \]
what enforces \( c_A = c_{jk} = c_{ik} \), i.e., the resulting bearing is a 4DOF bearing. Starting from two single spheres being the most simple 4DOF bearing, one can construct every other 4DOF bearing iteratively using the two presented rules for connecting as minimal requirements. The forming of additional contacts during connecting does not change the fact that the resulting bearing is a 4DOF bearing as long as it is bichromatic.

2. Practical ways of constructing complex 4DOF bearings

The space-filling bearing presented in Ref. [8] is also a 4DOF bearing, since in the space-filling limit every sphere is part of infinite many non-coplanar-4 loops, which are interconnected to each other through common contacts, what enforces \( c_{ij} = c \) for every contact. If one neglects all the spheres with a radius smaller than a lower cutoff \( r_{min} \), the bearings is also a 4DOF bearing for certain values of \( r_{min} \). For others one finds some spheres not being part of any loop which could be removed to end up with a 4DOF bearing. A sufficient condition to proof that it is a 4DOF bearing, is to show that the bearing consists of non-coplanar-4-loops which are throughout connected to each other, where two loops are called connected if they share at least two spheres. Another way to proof that the bearing is a 4DOF bearing would be to show that it can be constructed in the way explained in the previous section.

An algorithm to construct random bichromatic bearings that leads to non-coplanar-4-loops is shown in Ref. [9]. Given such a random bichromatic bearing, one can remove spheres that are not part of any loop and check if the remaining bearing is a 4DOF bearing. Using this method random 4DOF bearings that fit in a desired form can be constructed by giving constraints on the position of the spheres.

Appendix C: Blocking a single sphere

By applying a perturbation \( \Delta \vec{\omega}_i \) to a sphere \( i \) one can impose the changes \( \Delta_i \vec{B} \) and \( \Delta_i H \)
\[ \Delta_i \vec{B} = s_i m_i r_i \Delta \vec{\omega}_i \tag{C1} \]
and
\[ \Delta_i H = s_i m_i r_i \Delta \vec{\omega}_i \cdot \vec{x}_i. \tag{C2} \]
Remember that the origin of the position vector \( \vec{x}_i \) of the center of sphere \( i \) is the center of mass of the total bearing. Blocking a single sphere permanently and letting the bearing relax to a bearing state has the same effect on \( \vec{B} \) and \( H \) as applying a perturbation \( \Delta \vec{\omega}_i \) that leads to the sphere \( i \) being at rest in the bearing state, i.e., \( \Delta \omega_{i}^{bs} = \vec{0} \). To ensure \( \Delta \omega_{i}^{bs} = \vec{0} \) we find from
\[ \vec{\omega}_i^{bs} = \frac{s_i}{r_i} \left( \frac{\vec{B}_{new}}{M} + \frac{H_{new}}{I} \vec{x}_i \right) = \vec{0}, \tag{C3} \]
that
\[ \vec{B}_{new} = -MH_{new} \vec{x}_i/I. \tag{C4} \]
From the fact that \( \Delta_i \vec{B} \cdot \vec{x}_i = \Delta_i H \) (compare Eqs. (C1) and (C2)), we find using \( \Delta_i \vec{B} = \vec{B}_{new} - \vec{B}_{old} \) and \( \Delta_i H = H_{new} - H_{old} \), that
\[ H_{new} = \frac{H_{old} - \vec{B}_{old} \cdot \vec{x}_i}{1 + M|\vec{x}_i|^2/I}. \tag{C5} \]

Appendix D: Imposing a desired bearing state

We want to impose a desired change \( \Delta \vec{B} \) and \( \Delta H \) by applying external changes \( \Delta \vec{\omega}_i \) and \( \Delta \vec{\omega}_j \) to the angular velocities of sphere \( i \) and \( j \), respectively. We define \( \vec{\omega}_i = \omega_i \vec{\omega}_i \), where \( \vec{\omega}_i \) is the unit vector of the external change \( \Delta \vec{\omega}_i \). From Eqs. (C1) and (C2) we find
\[ \Delta \vec{B} = \Delta_i \vec{B} + \Delta_j \vec{B} = s_i m_i r_i \vec{\omega}_i + s_j m_j r_j \vec{\omega}_j \tag{D1} \]
and
\[ \Delta H = \Delta_i H + \Delta_j H = s_i m_i r_i \vec{\omega}_i \cdot \vec{x}_i + s_j m_j r_j \vec{\omega}_j \cdot \vec{x}_j. \tag{D2} \]
We multiply (dot product) Eq. (D1) with the position vector \( \vec{x}_j \) pointing from the center of mass of the bearing to the center of sphere \( j \) and subtract Eq. (D2) from it to find
\[ \omega_i = \frac{\Delta \vec{B} \cdot \vec{x}_j - \Delta H}{s_i m_i r_i \Delta \vec{\omega}_i \cdot \vec{x}_j}, \tag{D3} \]
which gives us the condition that \( \Delta \vec{\omega}_i \cdot \vec{x}_{ij} \neq 0 \). From Eq. (D3) we know \( \Delta \vec{\omega}_i \) since \( \Delta \vec{\omega}_i = \omega_i \Delta \vec{\omega}_i \) and we obtain using Eq. (D1) that
\[ \Delta \vec{\omega}_j = (\Delta \vec{B} - s_i m_i r_i \Delta \vec{\omega}_i)/(s_j m_j r_j). \tag{D4} \]

Appendix E: Determination of total mass

Applying a change \( \Delta \vec{\omega}_i \) to a sphere \( i \) will lead to a change \( \Delta \omega_{i}^{bs} \) of the angular velocity of sphere \( i \) between the previous to the new bearing state. We use
\[ \Delta \omega_{i}^{bs} = \frac{s_i}{r_i} \left( \frac{\Delta \vec{B}}{M} + \frac{\Delta H}{I} \vec{x}_i \right) \tag{E1} \]
and Eqs. (D1) and (D2) to find
\[ \Delta \omega_{i}^{bs} = m_i (\Delta \vec{\omega}_i / M + (\Delta \vec{\omega}_i \cdot \vec{x}_i) \vec{x}_i / I). \tag{E2} \]
Let us derive two scalar equations from Eq. (E2) by first squaring it (dot product) to find
\[ |\Delta \omega^{bs}_i|^2/m_i^2 = |\Delta \omega_i|^2/M^2 + 2(\Delta \omega_i \cdot \vec{x}_i)^2/(MI) + (\Delta \omega_i \cdot \vec{x}_i)^2/I^2 \]  
and second multiplying it (dot product) with \( \Delta \omega_i \) to find
\[ \Delta \omega^{bs}_i \cdot \Delta \omega_i/m_i = |\Delta \omega_i|^2/M + (\Delta \omega_i \cdot \vec{x}_i)^2/I. \]  
Combining the two we can eliminate the term \((\Delta \omega_i \cdot \vec{x}_i)^2\) and after some rearrangements we find
\[ |\Delta \omega_i^{bs}|^2/(m_i^2|\Delta \omega_i|^2) = 1/M^2 + (2/M + |\vec{x}_i|^2/I)(\Delta \omega^{bs}_i \Delta \omega_i/(m_i|\Delta \omega_i|^2) - 1/M). \] 
For a single change \( \Delta \omega_i \) and its induced change \( \Delta \omega^{bs}_i \) we see in Eq. (E5) that we have the two unknown quantities \( M \) and \( |\vec{x}_i|^2/I \). So with two changes \( \Delta_1 \omega_i \) and \( \Delta_2 \omega_i \) and their induced changes \( \Delta_1 \omega^{bs}_i \) and \( \Delta_2 \omega^{bs}_i \) one can find
\[ M = (B + \sqrt{B^2 - 4AC})/(2A), \] 
\[ A = |a_2 b_1 - a_1 b_2|, \] 
\[ B = |a_2 - a_1|, \] 
\[ C = |b_2 - b_1|, \] 
\[ a_n = |\Delta_n \omega^{bs}_i|^2/(m_i^2|\Delta_n \omega_i|^2), \] 
\[ b_n = \Delta_n \omega^{bs}_i \cdot \Delta_n \omega_i/(m_i|\Delta_n \omega_i|). \]  

Appendix F: Torsion dependence on bearing state parameters

For the bearing state, we can write the magnitude of the torsion torque \( |\vec{\tau}_{ij}| \) acting on sphere \( i \) due to contact with sphere \( j \) as an expression of \( \vec{B} \) and \( H \) by using
\[ \vec{\tau}_{ij} = \sigma (\vec{\omega}_j^{bs} - \vec{\omega}_i^{bs}) \cdot \hat{x}_{ij}, \] 
and the expression
\[ \vec{\omega}_i^{bs} = s_i \left( \frac{\vec{B}}{M} + \frac{H}{T} \vec{x}_i \right), \]

to find
\[ |\vec{\tau}_{ij}| = \left| \sigma \left( \left( \frac{1}{r_j} + \frac{1}{r_i} \right) \frac{\vec{B}}{M} + \left( \frac{\vec{x}_j}{r_j} + \frac{\vec{x}_i}{r_i} \right) \frac{H}{T} \right) \cdot \hat{x}_{ij} \right|. \]  
The torsion torques are zero for a single contact as long as
\[ \left( \left( \frac{1}{r_j} + \frac{1}{r_i} \right) \frac{\vec{B}}{M} + \left( \frac{\vec{x}_j}{r_j} + \frac{\vec{x}_i}{r_i} \right) \frac{H}{T} \right) \times \hat{x}_{ij} = 0, \]  
which in a bearing in general can not be fulfilled for all contacts at the same time, except when all spheres are at rest, i.e. \( \vec{B} = \vec{0} \) and \( H = 0 \). We define the total torsion as \( \tau = \sum (|\vec{\tau}_{ij}| + |\vec{\tau}_{ji}|) \), where the sum runs over all contacts. For a given value of \( \vec{B} \neq \vec{0} \) (or \( H \neq 0 \)) there is at least one value of \( H \) (or \( \vec{B} \)) in general different from zero that globally minimizes the total torsion. Figure 6 shows the isolines of the total torsion for a manually constructed 4DOF bearing against \( B_x \) and \( H \), where \( \vec{B} = (B_x, 0, 0) \).
Figure 6. (color online) Isolines of the total torsion defined as $\tau = \sum |\vec{\tau}_{ij}| + |\vec{\tau}_{ji}|$ against $H$ and $B_x$ in the bearing state of a manually constructed 4DOF bearing, where the sum runs over all contacts, $|\vec{\tau}_{ij}| = |\sigma(\vec{\omega}_j - \vec{\omega}_i) \cdot \hat{x}_{ij}|$, and $\vec{B} = (B_x, 0, 0)$. Only for the trivial case where $H = 0$ and $\vec{B} = \vec{0}$ we find no torsion friction since all spheres are at rest. Solid lines along $\partial \tau / \partial H = 0$ and $\partial \tau / \partial B_x = 0$ show minimized torsion with respect to $H$ and $B_x$, respectively. Radii of spheres between 0.08 and 0.15, density $\rho = 1$, $M = 0.272$, $I = 0.0474$. In total 42 spheres with 57 contacts. Torsion friction coefficient $\sigma = 0.01$. Isolines are equidistant since $\vec{\tau}_{ij}(a \cdot \vec{B}, a \cdot H) = a \cdot \vec{\tau}_{ij}(\vec{B}, H)$ for any $a \in \mathbb{R}$.