

Bridge Percolation

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In bridge percolation one gives a special weight to bridges, i.e., bonds that if occupied would create the first spanning cluster. We show that, for $p > p_c$, even far away from the critical point of classical percolation, the set of bridge bonds is fractal with a fractal dimension $d_{BB} = 1.215 \pm 0.002$. This new percolation exponent is related to various different models like, e.g., the optimal path in strongly disordered media, the watershed line of a landscape, the shortest path of the optimal path crack, and the interface of the discontinuous-percolation clusters. Suppressing completely the growth of percolation clusters by blocking bridge bonds, a fracturing line is obtained splitting the system into two compact clusters. We propose a theta-point-like scaling between this fractal dimension and $1/\nu$, at the classical-percolation threshold, and disclose a hyperscaling relation with a crossover exponent. A similar scenario emerges for the cutting bonds. We study this new percolation model up to dimension six and find that, above the upper-critical dimension of classical percolation, the set of bridge bonds is dense and has the dimension of the system.

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Percolation, the paradigmatic description of connectivity in random media, is relevant to a wide range of applications [1]. In the classical, random model, an infinite cluster is formed by connecting tenuous pieces with *cutting bonds*, yielding a continuous transition. There exist, however, situations where incipient spanning is on purpose suppressed to avoid breakthrough, like economic measures to avoid the bursting of a bubble or the rapid raising of emergency dikes to withhold a high flood. In those cases, the percolation threshold is delayed and, once it happens, can be rather abrupt even leading to a discontinuous transition [2, 3]. The question emerges on how to incorporate these constraints appropriately into the standard percolation model.

Closely before percolating, a configuration has sites or bonds that we call *bridges* (or *anti-red bonds*) which, once occupied, would create a spanning cluster. Here, we propose to randomly occupy bonds under the constraint that these bridge bonds have a different occupation probability and focus, in particular, on the special case where these bonds are never occupied, i.e., spanning is completely suppressed. We address how the set of these bridges scales with system size and also the properties of the resulting model. A remarkable by-product of this novel approach is the description of the bridge-bond set, at $p = p_c$, analogously to a theta point [4], i.e., for all values of p above p_c , it has a fractal dimension d_{BB} , while exactly at p_c it is $1/\nu$. We introduce a new tricritical crossover exponent, which we study up to dimension six, the upper-critical dimension of critical percolation. The proposed model assumes that global information is available at every local bond to control the bridges and suppress spanning. Models with global feedback mechanisms have been studied, for example, in social and economic systems, where an external observer can influence

the system [5], as well as in magnetic systems where a demagnetizing field emerges due to the dipole-dipole interactions [6].

The statistical weight of a percolation configuration in which a site or bond is occupied with probability p is $p^O(1-p)^E$ being O and E the number of occupied or empty sites or bonds, respectively. Let us now generalize this model by giving to the bridges a probability $p_1 \leq p$ of being occupied. For a nonpercolating configuration, the statistical weight becomes $p^O(1-p)^E(1-p_1)^{N_{BB}}$, where N_{BB} is the number of bridges. In fact, one could choose $p_1 = p^n$ which would mean that a bridge is only occupied after the n -th attempt, where $n = 1$ retrieves the classical case. Physically, a small p_1 describes a situation in which systematically spanning is suppressed and if $p_1 = 0$ the bridge bonds are never occupied, preventing the system to percolate. Furthermore, in the limit $p \rightarrow 1$ all bonds – except the bridges – are occupied and two compact clusters are obtained, separated, in $2D$, by a line which we call bridge-bond line (see Fig. 1). If $p_1 \ll L^{-d}$, the bridge bonds are the last ones to be occupied and the percolation transition becomes discontinuous (see *Supplemental Material* [7]).

As conjectured by Coniglio [8] (for a proof see *Supplemental Material* [7]), for classical percolation, $p_1 = p$, at the percolation threshold, both the set of bridge bonds and the set of cutting bonds are fractals of dimension $1/\nu$, where ν is the correlation length exponent. This result was numerically verified by Scholder [9]. For bridge percolation with $p_1 = 0$, since connectivity is suppressed, cutting bonds merge towards the bridge-bond line. In a different context, Cieplak, Maritan, and Banavar [10] investigated this line, finding that it is fractal with a fractal dimension of $d_{BB} = 1.22$. In fact, this line is in the universality class of the optimal path in a disordered system

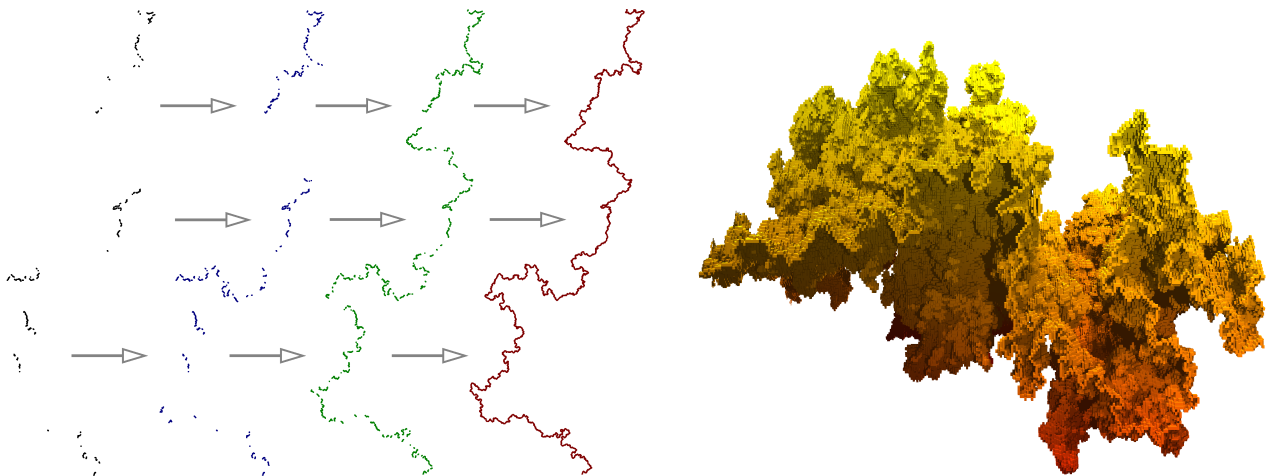


FIG. 1. Snapshot of the set of bridge bonds in two (line) and three (surface) dimensions. For 2D four stages have been included (from left to right): $p = p_c$ (black), $p = 1.01p_c$ (blue), $p = 1.05p_c$ (green), and $p = 1$ (red), while for 3D only the final set is shown. We considered in 2D a square lattice with 1024^2 sites and, in 3D, a simple-cubic lattice with 512^3 sites. The fractal dimension is $d_{BB} = 1.215 \pm 0.002$, in 2D, and $d_{BB} = 2.498 \pm 0.005$, in 3D.

[10] and the shortest path in trapping invasion percolation [11]. Recently, similar exponents have also been observed in different systems [12], namely, the shortest path of the optimal path crack [13], the watershed line [14], and the perimeter of the infinite cluster in discontinuous percolation [3].

To implement $p_1 = 0$, we occupy bonds sequentially, like in the classical case, under the constraint that, when leading to a percolating cluster – connecting opposite borders – they are identified as *bridge bonds* and remain unoccupied. We performed simulations on square lattices. Figure 2 shows the dependence of the number of bridge bonds N_{BB} on system size, for different values of the fraction of occupied bonds, namely, $p = p_c = 0.5$, $p = 0.51$, and $p = 0.8$. As expected, at the percolation threshold of classical percolation ($p = p_c$), the number of bridge bonds diverges with system size as $N_{BB} \sim L^{1/\nu}$, where ν is the correlation length exponent, with $\nu = 4/3$ in 2D; while for $p = 0.8$, $N_{BB} \sim L^{d_{BB}}$, with $d_{BB} = 1.215 \pm 0.002$. The latter is in fact observed at any value of $p > p_c$. However, for values of $p \gtrsim p_c$, like $p = 0.51$ we can observe, as depicted in Fig. 2, a crossover between the two different regimes. The inset of Fig. 3 shows N_{BB} , rescaled by $L^{d_{BB}}$, as a function of p , for different system sizes. The number of bridge bonds grows with the fraction of occupied bonds p , such that, $N_{BB} \sim (p - p_c)^\zeta$, where $\zeta = 0.50 \pm 0.03$ is a new exponent, which we call bridge-growth exponent. The overlap of the different curves confirms that the fractal dimension of the bridge bonds above p_c is d_{BB} , for all $p > p_c$. This result differs from classical percolation where, above p_c , bridge bonds are only observed for finite systems [8].

Analogous to the theta point of polymers [4], where the competition between attractive and repulsive forces leads to a crossover between two fractal dimensions, also

in bridge percolation the fractal dimension of the bridge bonds has a crossover from $1/\nu$, at $p = p_c$, to d_{BB} , above p_c . For the crossover scaling we verify the following *ansatz*,

$$N_{BB} = L^{1/\nu} \mathcal{F}[(p - p_c)L^\theta], \quad (1)$$

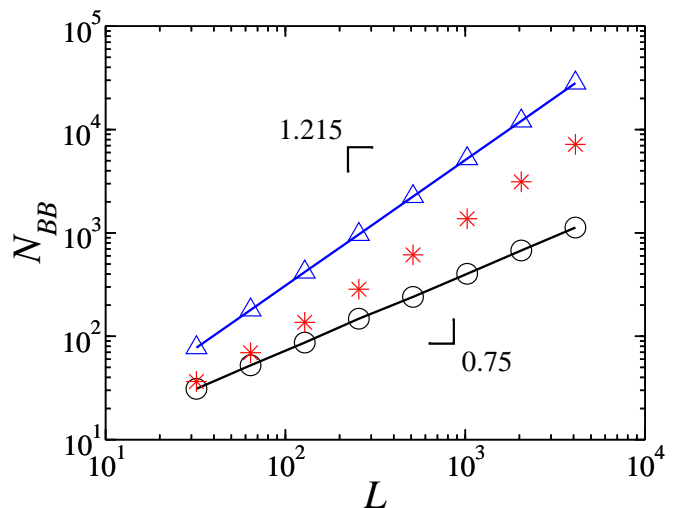


FIG. 2. Size dependence of the number of bridge bonds, N_{BB} , for $p_1 = 0$ and different values of p , namely, $p = p_c = 0.5$ (circles), $p = 0.51$ (stars), and $p = 0.8$ (triangles). The solid lines stand for the best fit. At $p = p_c$, as conjectured by Coniglio [8], $N_{BB} \sim L^{1/\nu}$, where $\nu = 4/3$ is the critical exponent of the correlation length in 2D. For $p > p_c$, the number of bridge bonds scales with $L^{d_{BB}}$. A crossover between the two regimes in system size is observed (stars) for values of p in the neighborhood of p_c . Systems of size L^2 have been considered, with L ranging from 32 to 4096. All results have been averaged over 10^4 samples. Error bars are smaller than the symbol size.

TABLE I. Values of the exponents ζ and φ up to dimension six. With increasing dimension, the ζ exponent converges to $\zeta = 1.5 \pm 0.7$ and φ goes to zero, revealing that the set of bridge bonds is dense, with the same dimension as the system.

d	ζ	φ
2	0.50 ± 0.03	0.785 ± 0.002
3	1.0 ± 0.1	0.502 ± 0.005
4	1.3 ± 0.5	0.26 ± 0.08
5	1.4 ± 0.6	0.1 ± 0.2
6	1.5 ± 0.7	0.0 ± 0.1

where $\mathcal{F}[x] \sim x^\zeta$ for $x \neq 0$, and is nonzero at $x = 0$; the exponent θ is the crossover exponent. Therefore, the following hyperscaling relation is obtained,

$$\theta = \zeta^{-1} \left(d - \varphi - \frac{1}{\nu} \right), \quad (2)$$

where $\varphi = d - d_{BB}$. In the main plot of Fig. 3 we see, for a 2D system, the scaling given by Eq. (1), with $\theta = 0.93$.

To study the dependence of both exponents ζ and φ on the spatial dimension, we analyze the same problem up to dimension six. In the *Supplemental Material* [7] we show the crossover scaling for bridge percolation on a simple-cubic lattice. Above p_c , the set of all bridge bonds has a fractal dimension $d_{BB} = 2.498 \pm 0.005$ and grows with $\zeta = 1.0 \pm 0.1$. Figure 4 shows the size dependence of N_{BB} , in the limit $p = 1$, for systems with size L^d , where $2 \leq d \leq 6$ is the dimension of the system. In the inset, we plot the exponent φ as a function of d . Since the set of bridge bonds blocks connectivity from one side

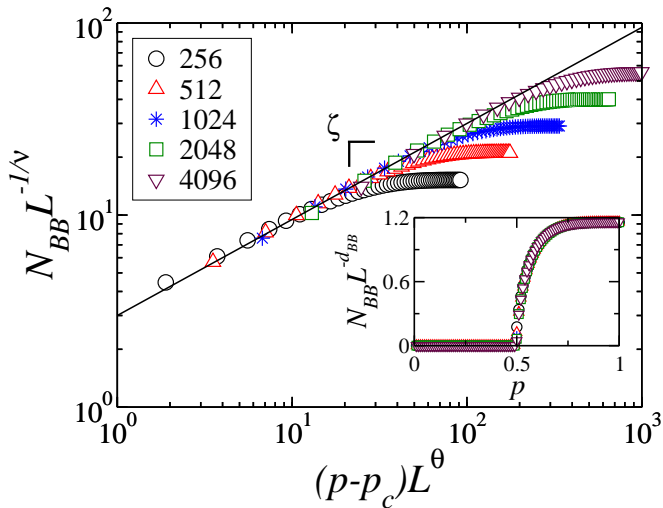


FIG. 3. Number of bridge bonds, N_{BB} , as a function of the fraction of occupied bonds, p , for 2D systems with different sizes $L = \{256, 512, 1024, 2048, 4096\}$ and $p_1 = 0$. The scaling function given by Eq. (1) is applied, with $\theta = 0.93$, obtaining $\zeta = 0.50 \pm 0.03$. In the inset, N_{BB} has been rescaled by $L^{d_{BB}}$, where $d_{BB} = 1.215$. All results have been averaged over 10^4 samples.

of the system to the other, its fractal dimension must follow $d - 1 \leq d_{BB} \leq d$, i.e., $0 \leq \varphi \leq 1$. With increasing dimension φ decreases. At the upper-critical dimension of percolation, $d_c = 6$, $\varphi = 0.0 \pm 0.1$ and the set of bridge bonds becomes dense having the same dimension as the system. Table I summarizes the exponents for dimensions 2 to 6. The bridge-growth exponent grows with d and converges to $\zeta = 1.5 \pm 0.7$ at the upper-critical dimension. For $d > 6$, above the critical dimension of percolation, the exponent φ remains zero and the dimension of the set of bridges is equal to d . This can be understood from the fact that above d_c one has an infinity of spanning clusters and thus many more possible bridges. Since the dimension of the set of bridges at $p = p_c$ is $1/\nu$ and above p_c is d , the crossover exponent increases with d , with the relation given by Eq. (2), where $\varphi = 0$.

Since one can interchange occupied and empty bonds, there exists a symmetry between bridges and cutting bonds, so one can raise the question of what happens when bonds are removed from a percolating system with the constraint that connectivity cannot be broken. Initially all bonds are occupied and at the end, since cutting bonds are never removed, a final cutting-bond line is obtained. Generalizing, if cutting bonds are removed with probability p_2 , then the statistical weight of a percolating configuration is $p^O (1-p)^E (1-p_2)^{N_{CB}}$, where N_{CB} is the number of cutting bonds. In the *Supplemental Material* [7] we show, for $p_2 = 0$, the size dependence of the number of cutting bonds N_{CB} for different values of p on a square lattice. Above p_c , as in the classical case, the

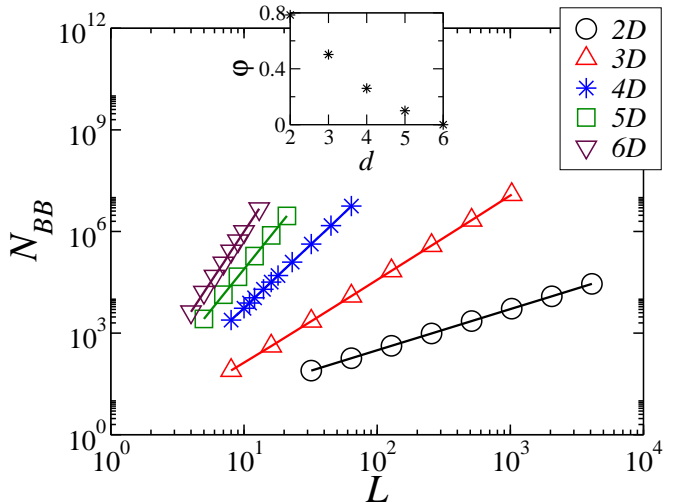


FIG. 4. Size dependence of the number of bridge bonds, N_{BB} , in the limit $p = 1$, up to dimension six. The solid lines stand for the best fit. All results have been averaged over 10^4 samples. Error bars are smaller than the symbol size. The inset shows the dependence on the spatial dimension of the exponent φ . At the upper-critical dimension of percolation, $d = 6$, the set of bridge bonds becomes dense having the dimension of the system.

percolation cluster is compact and there are no cutting bonds. For $p < p_c$, the set of cutting bonds is fractal with the same fractal dimension as the bridge-bond set, $d_{CB} = d_{BB}$, whereas at p_c it is $1/\nu$. For the crossover scaling a similar *ansatz* of the one given by Eq. (1) is verified, where the argument of the scaling function is then $(p_c - p)L^\theta$. The same hyperscaling of Eq. (2) is obtained with $\varphi = d - d_{CB}$. At $d = 2$, our numerical results corroborate the hypothesis of the same value of ζ and φ , for cutting and bridge bonds (see *Supplemental Material* [7]). For $d > 2$, the set of cutting bonds is a line with $d_{CB} \leq 2$ and the one of bridge bonds has a dimension above $d - 1$ so that the fractal dimensions differ. In fact, the cutting-bond line is a fractal line with the fractal dimension of the optimal path in the strong disorder limit [10]. Above the critical dimension, the set of cutting bonds has dimension two – like the shortest path at p_c [15] – for any value of $p \leq p_c$ and, therefore, no transition is observed.

The behavior of bridge bonds suggests that percolation-like models, with delayed threshold, lead to a fractal interface of the largest cluster with dimension d_{BB} . For example, Araújo and Herrmann [3] recently proposed two novel discontinuous percolation models – characterized by discontinuous transitions – where the growth of the largest cluster is systematically suppressed. For these models, the percolation threshold is shifted to larger fractions of occupied bonds and the obtained clusters are compact. Their interface, however, is self-similar with a fractal dimension consistent with d_{BB} .

In summary, suppressing connectivity (disconnection) between opposite borders leads to a fractal set of bridge (cutting) bonds, with the same fractal dimension even far away from the critical point of classical percolation. Bridge bonds have the same fractal dimension as the watershed [14], the optimal path crack [13], and the surface of discontinuous percolation clusters [3], while cutting bonds have the same dimension as the optimal path in strongly disordered media [10, 11]. In $2D$, there is a duality between bridges and cutting bonds and $d_{BB} = d_{CB}$, whereas for $d > 6$, $d_{BB} = d$ and $d_{CB} = 2$. From the above, we conjecture that the upper-critical dimension of the optimal path and the watershed is also $d_c = 6$. Finally, we show that, at the percolation threshold of classical percolation, bridge percolation displays a theta-point-like crossover.

This work opens up several challenges. Besides the need for more precise values of the bridge-growth exponent, and exact results in the mean-field limit, it would also be interesting to study the novel set of exponents in other related problems with different universality classes like, e.g., the Kasteleyn-Fortuin clusters in the q -state Potts model [8, 16] with or without magnetic field [17]. Regarding the Kasteleyn-Fortuin theorem, is also of interest, to search for the Hamiltonian which would lead

to the additional factor of $(1 - p_1)$ in the weight of each configuration, which is related to a global effect. For the cutting bonds, the study of the crossover for higher dimensions is another computational challenge. Finally, it would also be interesting to try to find the third-scaling field of our theta-like point.

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- [1] S. R. Broadbent and J. M. Hammersley, *Proc. Cambridge Philos. Soc.*, **53**, 629 (1957); D. Stauffer, *Phys. Rep.*, **54**, 1 (1979); D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, London, 1994); M. Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).
- [2] S. S. Manna and A. Chatterjee, *Physica A*, **390**, 177 (2011); J. Nagler, A. Levina, and M. Timme, *Nat. Phys.*, **7**, 265 (2011); W. Chen and R. M. D'Souza, *Phys. Rev. Lett.*, **106**, 115701 (2011); A. A. Moreira *et al.*, *Phys. Rev. E*, **81**, 040101(R) (2010); J. S. Andrade Jr. *et al.*, *ibid.*, **83**, 031133 (2011); N. A. M. Araújo *et al.*, *Phys. Rev. Lett.*, **106**, 095703 (2011); R. A. da Costa *et al.*, *Phys. Rev. Lett.*, **105**, 255701 (2010); P. Grassberger *et al.*, arXiv:1103.3728.
- [3] N. A. M. Araújo and H. J. Herrmann, *Phys. Rev. Lett.*, **105**, 035701 (2010).
- [4] P.-G. de Gennes, *Scaling concepts in polymer physics* (Cornell Univ. Press, Ithaca, 1979); I. Chang and A. Aharony, *J. Phys. I*, **1**, 313 (1991); P. H. Poole, A. Coniglio, N. Jan, and H. E. Stanley, *Phys. Rev. B*, **39**, 495 (1989).
- [5] N. A. M. Araújo, J. S. Andrade Jr., and H. J. Herrmann, *PLoS ONE*, **5**, e12446 (2010).
- [6] H. L. Richards, M. A. Novotny, and P. A. Rikvold, *Phys. Rev. B*, **54**, 4113 (1996); P. Cizeau *et al.*, *Phys. Rev. Lett.*, **79**, 4669 (1997).
- [7] See Supplemental Material at.
- [8] A. Coniglio, *Phys. Rev. Lett.*, **62**, 3054 (1989).
- [9] O. Scholder, *Int. J. Mod. Phys. C*, **20**, 267 (2009).
- [10] M. Cieplak, A. Maritan, and J. R. Banavar, *Phys. Rev. Lett.*, **72**, 2320 (1994); **76**, 3754 (1996).
- [11] M. Porto *et al.*, *Phys. Rev. Lett.*, **79**, 4060 (1997); *Phys. Rev. E*, **60**, R2448 (1999).
- [12] J. S. Andrade Jr. *et al.*, *Comput. Sci. Eng.*, **13**, 74 (2011).
- [13] J. S. Andrade Jr. *et al.*, *Phys. Rev. Lett.*, **103**, 225503 (2009); E. A. Oliveira *et al.*, *Phys. Rev. E*, **83**, 046113 (2011).
- [14] E. Fehr *et al.*, *J. Stat. Mech.*, P09007 (2009); *Phys. Rev. Lett.*, **106**, 048501 (2011).
- [15] S. Havlin and R. Nossal, *J. Phys. A: Math. Gen.*, **17**,

- L427 (1984).
- [16] C. M. Fortuin and P. W. Kasteleyn, *Physica*, **57**, 536 (1972); F. Y. Wu, *J. Stat. Phys.*, **18**, 115 (1978); A. Coniglio and W. Klein, *J. Phys. A: Math. Gen.*, **13**, 2775 (1980); Y. Deng, H. W. J. Blöte, and B. Nienhuis, *Phys. Rev. E*, **69**, 026123 (2004).
- [17] J. Kertész, *Physica A*, **161**, 58 (1989); Ph. Blanchard *et al.*, *J. Phys. A: Math. Theor.*, **41**, 085001 (2008); S. Biswas, A. Kundu, and A. K. Chandra, *Phys. Rev. E*, **83**, 021109 (2011).

Supplemental Material: Bridge Percolation

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BRIDGE BONDS

For classical percolation, corresponding to the case $p_1 = p$ in bridge percolation, the next bond to be occupied is randomly selected from the set of available ones. At a certain fraction of occupied bonds spanning occurs, namely, a cluster of connected sites touching opposite borders of the system emerges. Before spanning, those yet unoccupied bonds which, once occupied, would lead to the first spanning cluster are defined as bridge bonds (or anti-red bonds). For a hypercubic lattice of dimension d and linear size L , one defines $R_L(p)$ as the probability of spanning at p . A bond, which is randomly selected to be occupied ($p \rightarrow p + \delta p$), is a bridge bond with probability N_{BB}/N_{Av} , where N_{Av} is the number of available bonds, i.e., $N_{Av} = (1 - p)dL^d$. Since δp corresponds to a single bond then $\delta p = 1/(dL^d)$. The spanning probability is then,

$$R_L(p + \delta p) = R_L(p) + \frac{N_{BB}}{(1 - p)}\delta p. \quad (1)$$

Therefore, for large system sizes, as previously proposed, without prove, by Coniglio [1],

$$N_{BB} = (1 - p)\frac{dR_L(p)}{dp}. \quad (2)$$

From the behavior of $R_L(p)$ [2], one obtains at $p = p_c$, for any dimension, $N_{BB} \sim L^{1/\nu}$, as numerically verified by Scholder [3].

CUTTING BONDS

In classical percolation, one can reach the same configuration either by adding bonds to an initially empty system or by removing bonds from a configuration where all bonds are occupied. The state of the system solely depends on the actual fraction of occupied bonds. At a certain fraction of occupied bonds, when one bond is removed, the last spanning cluster is broken. Before breaking connectivity, in the set of occupied bonds, cutting bonds (red bonds) may exist, defined as bonds which, once removed, will break the last spanning cluster. The probability of breaking connectivity by removing a single bond ($p \rightarrow p - \delta p$) is the fraction of occupied bonds which are cutting bonds N_{CB}/N_{Oc} , where N_{CB} is the number of cutting bonds and N_{Oc} of occupied ones, i.e.,

$N_{Oc} = pdL^d$. If bonds are sequentially removed from the system, the spanning probability $R_L(p)$ changes as,

$$R_L(p - \delta p) = R_L(p) - \frac{N_{CB}}{p}\delta p. \quad (3)$$

Therefore, for large system sizes, as was also demonstrated by Coniglio [4] using a bond diluted percolation model,

$$N_{CB} = p\frac{dR_L(p)}{dp}. \quad (4)$$

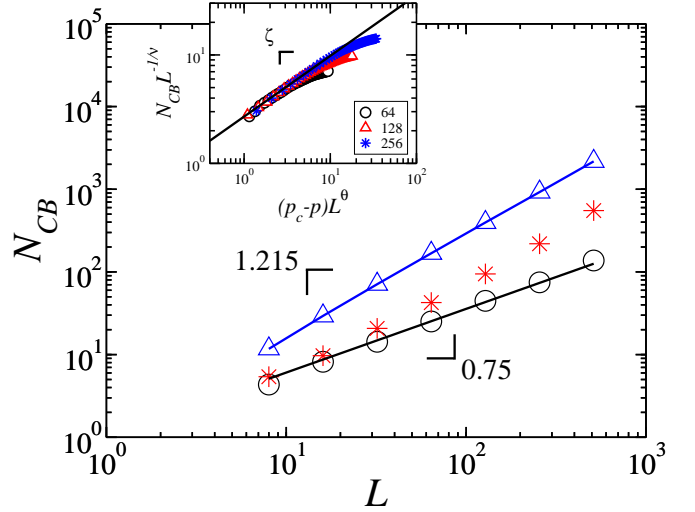


FIG. 1. Bonds are randomly selected and removed from the system except the cutting bonds. Size dependence of the number of cutting bonds, for $p_2 = 0$, N_{CB} , for different values of the fraction of cutting bonds in the system p , namely $p = p_c = 0.5$ (circles), $p = p = 0.49$ (stars), and $p = 0.2$ (triangles). The solid lines are guides to the eye. At $p = p_c$, $N_{CB} \sim L^{1/\nu}$, where $\nu = 4/3$ is the critical exponent related with the correlation length. For $p < p_c$, the number of cutting bonds scales with $L^{d_{CB}}$. A crossover between the two regimes with the system size is observed (stars) for values of p in the neighborhood of p_c . Systems of size L^2 have been considered, with L ranging from 8 to 512. Results have been averaged over 10^3 samples for most system sizes and over 50 samples for the largest one. Error bars are smaller than the symbol size. The inset shows the number of cutting bonds, N_{CB} , as a function of occupied bonds, p , for two-dimensional systems with L ranging from 64 to 256 (averaged over 10^2 samples). The scaling is given by Eq. (1) in the Letter, where the argument is then $(p_c - p)L^\theta$, with $\theta = 0.93$. We obtain $\zeta = 0.56 \pm 0.08$, which is compatible with the one for bridge bonds.

The fractal dimension of the cutting and the bridge bonds is the same. Since $R_L(p)$, in the thermodynamic limit of classical percolation, is a step function, there are only cutting and bridge bonds at the percolation threshold.

In Fig. 1 we see the size dependence of the number of cutting bonds in $2D$ when bonds are removed from the system with the constraint that cutting bonds are never removed. As discussed in the Letter for bridge bonds, a crossover is observed at $p = p_c$, between two fractal dimensions. In $2D$, for $p < p_c$ the fractal dimension of the cutting bonds d_{CB} is the same as observed for the bridge bonds for $p > p_c$, whereas at p_c it is $1/\nu$.

DISCONTINUOUS PERCOLATION

While realizations of discontinuous percolation have already been discussed in the context of bootstrap percolation [5], for other growth models observed transitions have been, until recently, of continuous nature. With the work by Achlioptas, D'Souza, and Spencer [6], the possibility for the existence of a discontinuous transition was raised for the first time. They suggested that, when a best-of-two product rule is considered to select bonds, the transition becomes sharper and apparently discontinuous. Several studies followed this work, applying the same rule on different graphs [7–11] and shedding light on some mechanisms that could eventually lead to discontinuity [12–15]. However, more recent results have raised controversy about the nature of the transition. The first signs were given by a similar model proposed by da Costa *et al.* [16] which, when numerically solved in the mean-field limit, reveals a continuous transition. Though a different model, its similarities with the product rule raised the possibility of a continuous transition in the original model. Afterwards, a detailed study of the order parameter at the transition point of the product rule revealed no gap in the thermodynamic limit [17, 18] and in its histogram, despite being bimodal for finite sizes [19], the distance between the peaks seems to vanish for the infinite system [20]. Recently, Riordan and Warnke [21] proved that for Erdős–Rényi graphs the product-rule yields a continuous transition. However, several other models have been developed, either with global competition [17, 22] or control of the maximum size [14, 23], where robust signs of discontinuity have been reported.

For bridge percolation, a discontinuous transition emerges as a limiting case. As mentioned in the Letter, for $p_1 = 0$, spanning is totally suppressed and for $p \rightarrow 1$ there are only two macroscopic dense clusters in the system divided by the set of bridge bonds. If $p_1 \neq 0$ but, nevertheless, close to zero, namely, $p_1 \ll L^{-d}$, the first bridge bond is occupied after all the other non-bridge bonds have been occupied and the two clusters merge into the spanning cluster. The transition is then obviously discontinuous and the jump in the size of the largest

cluster corresponds to the fraction of sites in the smaller of the two clusters. This allows, for the first time, to formulate a discontinuous percolation transition as a limit of a statistical equilibrium model since, up-to now, all realizations of discontinuous percolation have been irreversible growth models and thus far from equilibrium.

CROSSOVER SCALING IN 3D

Figure 2 shows the crossover scaling for the number of bridge bonds N_{BB} on a simple-cubic lattice with different sizes. Bonds are occupied randomly except for the bridge bonds. We applied the scaling function proposed in Eq. (1) of the Letter with $\theta = 1.36$, giving $\zeta = 1.0 \pm 0.1$.

PERCOLATION OF BRIDGES

As discussed in the Letter, for $p_1 = 0$, the set of bridge bonds merges towards a single connected line. In Fig. 3 we see the size of the largest cluster of sites on the edge of bridge bonds M_{BB} , rescaled by $L^{d_{BB}}$, as a function of p , for different system sizes. The inset shows χ , defined as,

$$\chi = \frac{1}{L^{2d_{BB}}} \left[\sum_i s_i^2 - M_{BB}^2 \right], \quad (5)$$

where the sum runs over all clusters of sites on the edge of bridge bonds and s_i is their size. Alike percolation on

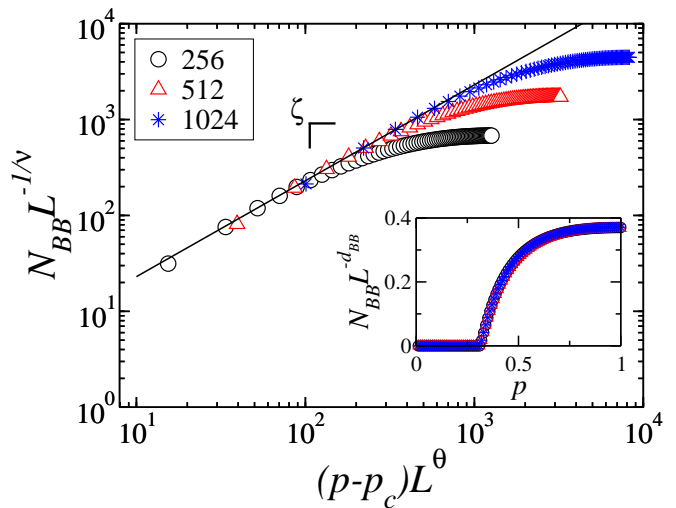


FIG. 2. Number of bridge bonds, N_{BB} , as a function of the fraction of occupied bonds, p , for three-dimensional systems (simple-cubic lattices) with different sizes $L = \{256, 512, 1024\}$. The scaling function proposed in this Letter is applied, with $\theta = 1.36$, obtaining $\zeta = 1.0 \pm 0.1$. In the inset, N_{BB} has been rescaled by $L^{d_{BB}}$, where $d_{BB} = 1.215$. All results have been averaged over 10^4 samples.

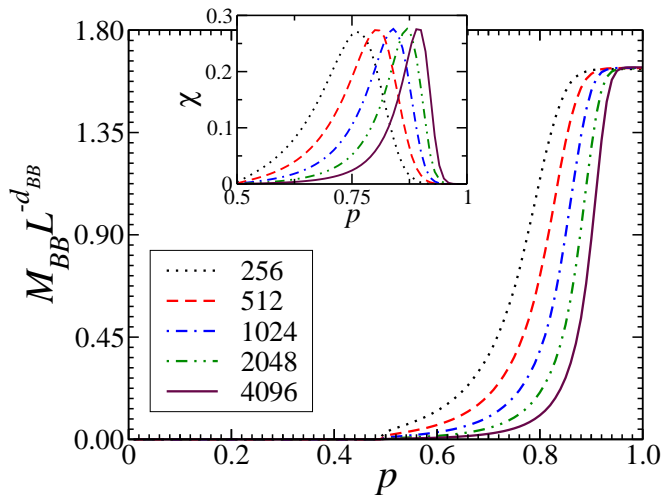


FIG. 3. Size of the largest cluster of sites on the edge of bridge bonds, M_{BB} , rescaled by $L^{d_{BB}}$, as a function of the fraction of occupied bonds p , for $p_1 = 0$. The inset shows χ defined by Eq. (5). Results have been obtained on a square lattice with L ranging from 256 to 4096. All results have been averaged over 10^4 samples.

a line, in the thermodynamic limit, the bridge-bond line only percolates at $p \rightarrow 1$.

- [1] A. Coniglio, Phys. Rev. Lett., **62**, 3054 (1989).
- [2] R. M. Ziff, Phys. Rev. Lett., **69**, 2670 (1992).
- [3] O. Scholder, Int. J. Mod. Phys. C, **20**, 267 (2009).
- [4] A. Coniglio, J. Phys. A: Math. Gen., **15**, 3829 (1982).
- [5] J. Adler, Physica A, **171**, 453 (1991).

- [6] D. Achlioptas, R. M. D'Souza, and J. Spencer, Science, **323**, 1453 (2009).
- [7] R. M. Ziff, Phys. Rev. Lett., **103**, 045701 (2009); Phys. Rev. E, **82**, 051105 (2010).
- [8] F. Radicchi and S. Fortunato, Phys. Rev. Lett., **103**, 168701 (2009); Phys. Rev. E, **81**, 036110 (2010).
- [9] Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim, Phys. Rev. Lett., **103**, 135702 (2009); Y. S. Cho, B. Kahng, and D. Kim, Phys. Rev. E, **81**, 030103(R) (2010).
- [10] H. D. Rozenfeld, L. K. Gallos, and H. A. Makse, Eur. Phys. J. B, **75**, 305 (2010).
- [11] R. K. Pan, M. Kivelä, J. Saramäki, K. Kaski, and J. Kertész, arXiv:1010.3171.
- [12] E. J. Friedman and A. S. Landsberg, Phys. Rev. Lett., **103**, 255701 (2009); H. Hooyberghs and B. V. Schaeferbroeck, Phys. Rev. E, **83**, 032101 (2011).
- [13] A. A. Moreira, E. A. Oliveira, S. D. S. Reis, H. J. Herrmann, and J. S. Andrade Jr., Phys. Rev. E, **81**, 040101(R) (2010).
- [14] N. A. M. Araújo and H. J. Herrmann, Phys. Rev. Lett., **105**, 035701 (2010).
- [15] J. S. Andrade Jr., H. J. Herrmann, A. A. Moreira, and C. L. N. Oliveira, Phys. Rev. E, **83**, 031133 (2011); N. A. M. Araújo, J. S. Andrade Jr., R. M. Ziff, and H. J. Herrmann, Phys. Rev. Lett., **106**, 095703 (2011).
- [16] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett., **105**, 255701 (2010).
- [17] J. Nagler, A. Levina, and M. Timme, Nat. Phys., **7**, 265 (2011).
- [18] H. K. Lee, B. J. Kim, and H. Park, arXiv:1103.4439.
- [19] L. Tian and D.-N. Shi, arXiv:1010.5990.
- [20] P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, arXiv:1103.3728.
- [21] O. Riordan and L. Warnke, arXiv:1102.5306.
- [22] S. S. Manna and A. Chatterjee, Physica A, **390**, 177 (2011).
- [23] W. Chen and R. M. D'Souza, Phys. Rev. Lett., **106**, 115701 (2011).