Impact of Perturbations on Watersheds

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We find that watersheds in real and artificial landscapes can be strongly affected by small, local perturbations like landslides or tectonic motions. We observe power-law scaling behavior for both the size-distribution of areas enclosed by the original and the displaced watershed as well as the probability density to induce, after perturbation, a change at a given distance. We find universal exponents for real and artificial landscapes, where in the latter case the exponents depend linearly on the Hurst exponent of the applied fractional Brownian noise. The obtained power-laws are shown to be independent on the strength of perturbation. Theoretical arguments relate our scaling laws for uncorrelated landscapes to properties of invasion percolation.

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Watersheds are the lines separating adjacent drainage basins (catchments) and play, hence, a fundamental role in water management [1], landslides [2, 3] and flood prevention [3, 4]. Since ancient times they have been used to delimit boundaries and have already become issues in disputes between countries [5]. Moreover, similar problems also appear in other areas such as Image Processing [6] and Medicine [7], which shows the generality and importance to fully understand the subtle dynamical properties of watersheds. But how sensitive are watersheds to slight localized modifications of the landscapes? Can these perturbations produce large, non-local changes in the watershed? Geographers and geomorphologists have studied the evolution of watersheds in time and found it to be driven by local events called stream capture or piracy, which occur due to erosion, natural damming, tectonic motion as well as volcanic activity [8–10]. These mechanisms were also investigated in small scale experiments [11]. Although these events are stated to be rare [12], they can be catastrophic [8] and do still occur today [9], eventually having a huge impact on the hydrologic system. Moreover, historical stream captures are shown to have affected the biogeography [13]. Recently the effect of tectonic uplift on channel-geometry was studied numerically [14]. A similar effect, called segmentation failure, is known in image processing [15], where much attention is payed on how to circumvent such failures.

In this Letter we investigate numerically and analytically the effects of topological modifications like landslides or tectonic motion onto the watershed. We show that these modifications can indeed trigger non-local effects that follow power-laws with the distance, generating quite dramatic changes of the watershed. For example, as shown in Fig. 1, after a local height change of less than 2 m at a location (cross), close to the Kashabowie Provincial Park, some kilometers North of the US-Canadian border, the watershed is displaced, such that the original (red) and the new watershed (blue) enclose an area $A \sim 3730 \text{ km}^2$. Here a model is then developed to provide a qualitative and quantitative description of this phenomenon.

In our simulations, we use real and artificial landscapes in the form of Digital Elevation Maps (DEM), consisting of discretized elevation fields. The discretization units we call sites. We define the drainage according to the steepest descent along the coordinate directions. For a given DEM we group all sites into two distinct sets, corresponding to two drainage basins (catchments), each draining to either one edge or the other of a chosen pair of opposite edges of the DEM. The watershed is then defined as the dividing line between the sites of the two catchments. For the determination of this line we use an iterative application of an invasion percolation procedure (IP) [16]. For a given landscape, as shown in Fig. 1, we initially determine its watershed (red line). Then a local event is induced by changing the height $h_k \rightarrow h_k + \Delta$ at a single site $k$ (cross in Fig. 1) of the DEM, where $\Delta$ is the perturbation strength. Applying IP on the perturbed landscape, a new watershed is found (blue line). The displacement of the watershed is quantified by measur-
ing the area $A$ between the original and the perturbed watershed. The water only escapes from that area at a single site, which we call outlet. Note that, in this procedure, two outlets are involved, one before ($i$-outlet) and another ($j$-outlet) after the perturbation is applied. As we choose $\Delta > 0$, the $i$-outlet, which is not at the original watershed, always coincides with the perturbed site $k$. The perturbation of the original landscape is repeated for every site of the landscape, so that each perturbed landscape deviates from the original only in one single site. We fix $\Delta = \Delta_w = |h_{\text{max}} - h_{\text{min}}|$, i.e. the height difference between the lowest $h_{\text{min}}$ and highest height $h_{\text{max}}$ of the landscape, corresponding to the largest perturbation that can make changes, which means that all possible changes are obtained. In all definitions hereafter, we consider only those perturbations which lead to a displacement of the watershed. We study in the following the size-distribution $P(A)$ of the areas $A$, the probability distribution $P(R)$ of the Euclidean distance $R$ between the two outlets, and the dependence between $A$ and $R$. For this, we define the average area $\langle A \rangle$ and the size-distribution $P(A|R)$ of areas $A$ associated with an outlet distance $R$.

First, we study several natural landscapes, from mountainous (e.g. Rocky Mountains) to rather flat landscapes (e.g. US-CAN, Kongo and Germany). The DEM data was obtained from the SRTM-project [17], where we used sizes of 2700 km $\times$ 2700 km (1080 km $\times$ 1080 km for Germany) and a resolution of 540m. As shown in Fig. 2, we find that the size-distribution of areas follows a power-law, $P(A) \sim A^{-\beta}$, with an exponent $\beta = 1.65 \pm 0.15$ for all landscapes. The probability distribution $P(R)$ of outlet distances $R$ also obeys a power-law, $P(R) \sim R^{-\rho}$, with $\rho = 3.1 \pm 0.3 \approx 2\beta$ (not shown). This implies $\langle A \rangle \sim R^2$, which agrees well with our data (inset of Fig. 2). The size-distribution for a given distance $R$ scales as $P(A|R) \sim A^{-\alpha}$ with $\alpha = 2.3 \pm 0.2$. Furthermore, the watershed remains invariant for perturbations located at distances $R > 100$ km.

![Fig. 2. The size-distribution $P(A)$ is shown for various regions: Rocky Mountains, Andes and Appalachian (unshifted); Brazil and Europe (shifted by a factor of 100 for better visibility); US-CAN, Kongo and Germany (shifted by 10’000). The solid line shows the best fit to the Andes data with a power-law of exponent $-1.65 \pm 0.15$. The inset shows $\langle A \rangle$ as a function of $R$ for the Rocky Mountains at resolutions $\sim 270$m (squares), $\sim 540$m (circles) and $\sim 1350$m (triangles). The solid line has slope 2.](image)

![Fig. 3. Data collapse of the distribution $P(R)$ for three different system sizes $L = 129$, 257, 513 (triangles, circles and squares). The line represents the best fit of a power-law to the data for the largest landscape (squares) revealing an exponent $\rho = 2.21 \pm 0.01$. The inset shows the size-distribution $P(A)$ of the areas for the same system sizes. The solid line represents the best fit to the data of a power-law with an exponent of $\beta = 1.16 \pm 0.03$.](image)

To understand these power-laws and the dependence between $A$ and $R$, we studied artificial landscapes, where the heights follow a uniform distribution in the absence of any spatial correlation. In Fig. 3, we present the results obtained for several system sizes, using the same procedure as for the natural landscapes. The probability density $P(R)$ again follows a power-law $P(R) \propto R^{-\rho}$. We estimate $\rho = 2.21 \pm 0.01$ using the scaling $P(R) = L^\rho f[R/L]$, where $L$ is the linear dimension of the landscape. For the size-distribution $P(A) = L^\beta f[A/L^2]$ we obtain an excellent data collapse for $\beta = 1.16 \pm 0.03$ (see inset of Fig. 3). In the case of the distribution $P(A|R)$ at fixed outlet distance, we again find a power-law $P(A|R) \sim A^{-\alpha}$ (not shown). Finite size scaling analysis yields an exponent $\alpha = 2.23 \pm 0.03$ independent on the value of $R$. Assuming that $R$ describes the extension of $A$ in every direction the relation $\rho = \alpha$ is reasonable. Indeed, this assumption seems to be well supported by the similarity of the obtained values of the exponents. The area $A$ was rescaled by $L^2$, indicating that the areas are compact. Indeed, considering finite size scaling carefully, we find $\langle A \rangle \sim R^2$ in perfect agreement with our data (not shown). Furthermore, the compactness of the areas is supported by the measured value $\beta = 1.16 \pm 0.03$ agreeing well with the re-
lution $\beta = \rho/2 \approx 1.11$. Next we show that we can match the exponents quantitatively by introducing long-range correlations, as present in real geological systems.

In the following we use fractional Brownian motion (fBm) on a square lattice [18] to incorporate long-range correlations controlled by the Hurst exponent $H$. The limit $H = -1$ corresponds to the uncorrelated case presented above. The exponents $\alpha$, $\beta$, and $\rho$ were calculated for several values of $H$ (see Fig. 4). As shown in Fig. 4, we observe that both $\beta$ and $\rho$ increase with $H$. Furthermore, the relationship $\beta = \rho/2$ is maintained, since the areas remain compact in the entire range of $H$ values. Around $H = -0.5$, $\alpha$ starts to deviate from $\rho$ and for $H > 0$ we observe $\alpha$ to decrease. Previously, we had assumed $R$ to reflect the extension of the area, i.e. the outlets to reside typically on opposite sides of the area. To check whether this is still valid, we measured the angle $\theta$ between the lines connecting the center of mass of the area with the two outlets. We observe the average angle to decrease as function of $H$ (see Fig. 4) and of the area $A$ at a fixed value of $H$ (not shown). This implies that, on average, the two outlets approach each other with increasing $H$ (see also insets of Fig. 4), so that $R$ is no longer representative of the area extension. Finally we find good quantitative agreement with the exponents obtained from the natural landscapes for $0.3 < H < 0.5$, which is the known range of Hurst exponents for real landscapes on length scales larger than 1 km (see Ref. [19] and references therein). Hence, we end up with a model providing a complete quantitative description of the effects observed on natural landscapes.

Next we analyze quantitatively the impact of the perturbation strength $\Delta$ on the watershed. In Fig. 5 the number of perturbed sites $N$ that change the watershed, depending on the perturbation strength $\Delta$ applied for uncorrelated (squares), Andes (triangles) and fBm landscapes with $H = 0.3$ (circles). The solid line shows the analytic result obtained from Eq. (1) for uncorrelated landscapes. The inset shows the average area $\langle A \rangle$ as function of the distance $R$ between the outlets for $\Delta/\Delta_w = 1$, 0.016 and 0.00025 (pluses, crosses, squares), and $L = 513$.

FIG. 4. The exponents $\alpha$ (squares), $\beta$ (circles) and $\rho$ (triangles) are shown for several values of the Hurst exponent $H$. Each point results from a similar study as done for the uncorrelated landscapes. The exponents for the natural landscapes (open symbols) match well for $0.3 < H < 0.5$. The average angle $\theta$ (in radians) between the outlets from the center of mass (crosses) is shown too. The insets depict schematic shapes of the areas and positions of the two outlets for small (left) and large (right) values of $H$.

For landscapes with uniformly distributed heights, we found $p_o(h)$ to be still a uniform distribution. Then we obtain with Eq. (1), $N(\Delta) = (h_w \Delta - \Delta^2/2)2N_{\text{max}}/(L^2h_w^2)$ (see Fig. 5), in perfect agreement with the data and being approximately linear for $\Delta < h_w$. The observed power-laws are maintained for all values of $\Delta$, as can be seen exemplary for $\langle A \rangle$ in the inset of Fig. 5. This brings us to the conclusion that infinitesimally small perturbations have qualitatively the same effect on the watershed as any larger perturbation strength $\Delta$.

In the case of uncorrelated landscapes, for a given area
A, the corresponding invasion percolation cluster is obtained by starting the penetration process from one outlet to another, always growing along the steepest descent. The area A can therefore be understood as the envelop of this IP-cluster. From percolation theory, the fractal dimension of the IP cluster is \( d_f = \frac{91}{48} \) in two dimensions [20], which implies that \( \langle A \rangle \sim M^{2/d_f} \), where \( M \) is the number of sites (mass) of the cluster. This result is consistent with our simulations. The size-distribution \( P(M|R) \) of IP-clusters between two sites at a fixed distance \( R \) is known to follow a power-law \( M^{-\alpha} \) with \( \alpha^* = 1.39 \) [20]. Note that, for comparison of our results to Araújo et al. [20], \( P(M|R) \) needs to be divided by \( M \), as we grow the IP-cluster starting from the outlet at the watershed, which is always the highest of the \( M \) sites of the cluster. Hence we expect \( P(M|R) \sim M^{-(\alpha^*+1)} \), what is indeed in good agreement with our data (not shown). We can now relate our exponent \( \alpha \) of the distribution of areas at fixed distance to \( \alpha^* \), which describes the size-distribution of IP-clusters, as \( P(A|R) = \langle A \rangle^{-\alpha} (M) \propto M^{2/d_f} \propto P(M|R) \). We obtain \( \alpha = \frac{d_f}{2}(\alpha^*+1) \approx 2.266 \), which is very close to what we measured (\( \alpha \approx 2.23 \pm 0.03 \)). Therefore, we can relate our results on uncorrelated landscapes to the sub-critical point to point invasion percolation [20] and to the mass distribution of avalanches that occur during the IP-cluster growth [21–23].

In summary, we were able to show that small and localized perturbations can have large impact on the shape of watersheds even at very large distances, hence having a non-local effect. The size-distribution of changes \( P(A) \) is found to decrease as a power-law with exponent \( \beta = 1.65 \pm 0.15 \) on all studied real landscapes from mountainous (e.g. Rocky Mountains) to rather flat (e.g. US-Canadian border), for perturbations within a distance \( R < 100 \) km. Applying perturbations to model landscapes with long-range correlations, we determined the dependence of the scaling exponents on the Hurst exponent, finding good quantitative agreement with real landscapes for \( 0.3 < H < 0.5 \). The obtained exponents \( \alpha, \beta \) and \( \rho \) are independent of the perturbation strength \( \Delta \). For uncorrelated landscapes we could derive a relation with invasion percolation. It is known that watersheds [16] on uncorrelated landscapes are related to “strands” in Invasion Percolation [24], random polymers in strongly disordered media [25], paths on MST’s [26], the backbone of the optimal path crack [27] and the cluster perimeter in explosive percolation [28]. Hence, our results can be potentially applied to all these problems.

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