Analytic study of snow failure using a fiber bundle model

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Abstract

Recently a modified fiber bundle model was introduced to investigate the avalanche initiation in dry-snow slabs, that is believed to start in a weak snow layer. Laboratory experiments showed, that beside fracture a healing mechanism (sintering) also occurs in this weak layer. We use an extended fiber bundle model to study this highly disordered material. We present a model in which every broken fiber has the possibility to regain strength. This sintering process is characterized by a sintering time \( T_s \), that is different from the characteristic breaking time. For simplicity we keep the upper and lower plates rigid, and by this the elongation of fibers is unequivocally defined by the displacement of the upper plate, which gives the possibility of analytical handling. The aim of this work is to calculate the constitutive equation of the model. We present a semi-analytic formula with which the simulation results are in perfect agreement. In special cases it is also possible to obtain the asymptotic behavior of the system in a closed form. These results could be used to get a deeper understanding of avalanche initiation.
I. INTRODUCTION

Understanding the initiation of avalanches in dry-snow slabs is an important problem. Previous studies imply, that such avalanches happen due to a weak snow layer below a solid snow slab [1]. It was shown by McClung [2] that the existence of such a weak layer is necessary to have avalanches in the case of dry-snow.

On a slope the gravitational force generates normal and shear deformation. After a certain time small cracks appear in the weak layer, while simultaneously some grains that touch each other may bond again, due to the fact, that the temperature is near to their melting point. These processes of breaking and bonding are competing against each other.

There are many different possible techniques to study this phenomena. The advantage to use a fiber bundle model (FBM) is that in spite of its simplicity, FBM captures the most important ingredients of the failure process and makes it also possible to obtain several important quantities in closed analytic form. To capture the shear deformation one should in principle use beams instead of fibers, but F. Raischel et al. showed [3], that with a proper choice of breaking thresholds and transformation formulas it is possible to achieve the same results with simple fiber based models. The work of I. Reiweger et al. [4] gave evidence that the simulations based on our model are in good agreement with laboratory experiments. It was found, that the system can behave in a brittle or in a ductile way depending on the temperature and the shear strain rate [1, 5]. The system behaves ductile at slower and brittle at faster rates.

II. MODEL

Laboratory experiments showed [5, 6] that the shear deformation is mostly concentrated in the weak layer. The model only tries to capture the behavior in this region of the dry-snow slab. The snow crystals, e.g. buried hoar crystals that are mostly constituting this layer are represented by fibers.

The fiber bundle model under consideration is composed of $N$ fibers assembled in parallel on a two dimensional square lattice. The fibers are considered to be linearly elastic until the breaking point with identical initial length and Young modulus. The disorder in the system is captured by random breaking threshold $\sigma_{i,\text{th}}, i = 1, \ldots, N$. The breaking threshold of a
single fiber is taken from a distribution characterized by a probability density function \( p(\sigma) \) and cumulative probability distribution function \( P(\sigma) \). The two widely used distributions are: the uniform distribution with density function

\[
p(x) = \frac{x}{x_{\text{max}} - x_{\text{min}}} \tag{1}
\]

and the Weibull distribution

\[
p(x) = \frac{\mu}{\lambda} \left( \frac{x}{\lambda} \right)^{\mu-1} e^{-\left( \frac{x}{\lambda} \right)^{\mu}} \tag{2}
\]

with scale factor \( \lambda \) and shape factor \( \mu \). In the original FBM the load sharing is also an important ingredient of the system, but in this case experiments and simulations are only carried out in a strain controlled way and so there is no load redistribution in the system. In the experiments, just like in the simulations the bottom snow layer is fixed, while the upper layer moves along the x-axis with a constant velocity \( v = \frac{\Delta x}{\Delta t} \). From this the equation of the global shear strain rate \( \dot{\varepsilon}_g \) is:

\[
\dot{\varepsilon}_g = \frac{1}{l_0} \frac{\Delta x}{\Delta t} \tag{3}
\]

where \( l_0 \) is the initial fiber length, and \( \Delta t \) is the time unit that is set to unity here. In the FBM the fibers are only capable to deal with uniaxial tension, therefore we must neglect the effect of bending and shear deformation. This is not a strong limitation, since as mentioned before snow is a highly brittle material which means that the deformation of a single fiber is very small. Since the upper plate is supposed to be rigid, the load on a single fiber is always proportional to the external deformation. The only question now is how to define the tension on a single fiber. We used two different methods. The first is less realistic, but gives simpler equations. In this case we assume, that the elongation of the fiber equals to the global displacement as long as the fiber is intact (see Eq. 4). This simpler case will allow us to get further in the analytic calculations. Here:

\[
\Delta l = t_i \frac{\Delta x}{\Delta t} = \dot{\varepsilon}_g t_i l_0 \tag{4}
\]

where \( t_i \) is the time, and \( t_i/\Delta t \) the number of time steps that the fiber has been intact. In the second case by using simple geometrical considerations a more realistic approach gives (see Fig. 1):

\[
\Delta l = \sqrt{l_0^2 + \left( \frac{t_i}{\Delta t} \Delta x \right)^2} - l_0 \tag{5}
\]
The force needed to deform an intact fiber is \( f_j(t) = a \sigma_j(t) = a E \varepsilon_j(t) \) where \( E, \sigma_j(t), a, \varepsilon_j(t) \) are the Young modulus, the stress, the cross-section, and the strain of the fiber, respectively. When the stress \( \sigma_j(t) \) acting on a fiber reaches its previously defined \( \sigma_{th,j} \) threshold then the fiber breaks immediately, and in the next time step the stress on the fiber drops down to zero. The difference with the well-known FBM, is that in our model after each time step each broken fiber gets the chance of re-bonding (sintering). The number of broken fibers equals to \( N - N_i \), where \( N_i \) is the number of intact fibers. The probability of sintering \( P_s \) depends on the number of broken fibers, and is written in the following form

\[
P_s = p_{\text{max}} (1 - n_i)^k
\]

where \( p_{\text{max}} \) is a parameter related to the physical properties of the snow and \( n_i = N_i/N \) is the fraction of intact fibers, from which follows, that \( 1 - n_i \) is the fraction of broken fibers. For simplicity we chose \( k = 1 \), though \( k = 2 \) might be more realistic. The calculations can be performed for both cases but in the case of \( k = 1 \) we only have to solve a quadratic equation. After a fiber sinters in the time step \( t_n \), a new strength threshold is chosen from the same probability distribution, and the fiber is considered to be intact at the next time step \( t_{n+1} = t_n + \Delta t \), but it only reaches its new strength completely after a certain \( T_s \) sintering time. The change of fiber strength during the sintering process is given by

\[
\sigma'_{th,j}(t) = \left( 1 - e^{-t/T_s} \right) \sigma_{th,j} \quad \text{for } t < T_s.
\]

where \( \sigma_{th,j} \) on the right side of the equation is the new strength threshold of the fiber.

The macroscopic behavior of the system is characterized by the constitutive law, for the global stress \( \sigma_g(t) \) defined as

\[
F(t) = \sum_{j=1}^{N_i} f_j(t), \quad \sigma_g(t) = \frac{F(t)}{A}
\]

where \( A \) is the area of the snow slab. This law is controlled by three parameters: the distribution of the thresholds, the sintering probability \( p_{\text{max}} \), and the \( T_s \) sintering time.

To understand the effect of the three parameters we developed a highly efficient algorithm of the presented model. We are able to simulate systems with size up to \( 10^7 \), with time complexity in the order of minutes, and so we get nice smooth curves (see curves by symbols on Figs. 3, 5).
III. MACROSCOPIC BEHAVIOR OF THE SYSTEM

In the classical FBM the constitutive relation for a deformation controlled system is given by $\sigma = E\varepsilon [1 - P(E\varepsilon)]$ [7, 8], where $P(E\varepsilon)$ is the cumulative probability distribution of the fiber strength thresholds. The term $E\varepsilon$ is the stress on a fiber, while the term in the brackets is the probability that a fiber survives, i.e. the fraction of the intact fibers in the system.

To write down the macroscopic behavior we have to first define the fraction of intact fibers, which is more difficult than in the classical FBM. The difficulty here compared to other healing models, is that once a fiber is broken, we have no information anymore about its past. If we examine all the possibilities that could happen to a broken fiber, we obtain the following semi-analytic equation of $n_i(t)$.

$$n_i(t) = [1 - P(E\varepsilon(t))] + \sum_{t'=1}^{t-1} P_s(n_i(t'), p_{max}) [1 - n_i(t')] [1 - P'(E\varepsilon(t-t'))]$$  \hspace{1cm} (9)

The first term is taken from the classical FBM. With time this part vanishes from the equation. The sum goes over all intact fibers in the system. The first part of it defines the fraction that was re-bonded in the time step $t'$, while the last term checks if they are still intact. Due to the sintering process the $P(E\varepsilon(t))$ and the $P'(E\varepsilon(t))$ probability distributions are not equal.

Since experiment and simulations are carried out in a deformation controlled way, the stress acting on a fiber is always proportional to the time elapsed since the fiber is intact. Using Eqs. (4), (5) and (7) we can define a minimum load bearing capacity $\sigma_{th,min}(t)$ that is needed to survive the $t$th step of the sintering process, where $t = 1, .., T_s - 1$. If the breaking threshold of a fiber is smaller than any value of $\sigma_{th,min}(t)$ than this fiber will fail during the sintering process. This critical threshold can be written in the following form:

$$\sigma_{th,min}(t) \geq \frac{E\varepsilon(t)}{1 - e^{-t/T_s}}$$  \hspace{1cm} (10)

On the left side of Fig. 2 we can see that $\sigma_{th,min}(t)|_{t=1} \geq \sigma_{th,min}(t)$, $t = 1, .., T_s - 1$, which implies that if a fiber can survive the first step of the sintering, it will survive the whole process. This means that $P'(\varepsilon(t))$ only differs from $P(\varepsilon(t))$ in the sense that a $T_s$ dependent percentage of the fibers breaks immediately. Considering this, the probability equation
becomes

\[ P'(E\varepsilon(t)) = P(E\varepsilon(T_s)) \quad t \leq T_s \]  
\[ P(E\varepsilon(t)) \quad t \geq T_s \] (11) (12)

Using Eq. (5) to define the elongation of fibers (see Fig. 2 right), the minimal strength threshold monotonically increases so there is no guarantee, that if a fiber survived a certain step of the sintering process it will reach its maximum strength threshold. In this case it is possible to define \( P' \) in the following way:

\[ P'(E\varepsilon(t)) = P(\sigma_{th,min}(t)) = P \left( \frac{E\varepsilon(t)}{1 - e^{-\frac{t}{T_s}}} \right) , \ t \leq T_s \] (13)
\[ P(E\varepsilon(t)) , \ t \geq T_s \] (14)

Using these modified probability distribution functions in Eq. (9) the semi-analytic formula matches the simulation results perfectly (see Fig. 3 left). From Eq. (9) it is straightforward to write down the constitutive equations. This solution is also a recursive semi-analytic function:

\[
\sigma_g(t) = \left[ 1 - P(E\varepsilon(t)) \right] \varepsilon(t) + \sum_{t'=1}^{t-1} P_s(n_i(t'), p_{max}) \left[ 1 - n_i(t') \right] \left[ 1 - P'(E\varepsilon(t-t')) \right] \varepsilon(t-t')
\] (15)

The simulation results (curves with symbols) on the right of Fig. 3 are in a perfect agreement with this equation (solid lines). This semi-analytic formula can deal with both distributions and with both the elongation calculating methods (Eq. (4) and Eq. (5)).

IV. ASYMPTOTIC BEHAVIOR

From Fig. 3, we see that breaking and healing are competing with each other and after a while reach a steady state. We can also calculate the asymptotic behavior of the curves in a closed analytic form. First we study how the strength distribution of the fiber changes in the sintering process. We found that as long as the system is not in the steady state the distribution is changing continuously, but when the system reaches the dynamic equilibrium, the distribution is also in a steady state. In the simplest case, when we have a uniform distribution, and the elongation of a fiber is given by Eq. (4), and \( k = 1 \) in Eq. (6)
the constant density function of the uniform distribution becomes a linear density function (see Fig. 4). The circles show the original probability density function, while the triangles show it in the steady state. What we know is that the integral of the curve drawn with triangles equals to the fraction of intact fibers in the steady state. Using this we are able to write down the equation for the density function:

\[ f(\varepsilon) = \frac{2 \left[ n - (1 - n)^2 p_{\text{max}} \frac{\varepsilon_{\text{th},\text{min}}}{\varepsilon_{\text{max}}} \right]}{\varepsilon_{\text{max}}^2 - \varepsilon_{\text{th},\text{min}}^2} \]  \hspace{1cm} (16)

where \( \varepsilon_{\text{th},\text{min}} = \sigma_{\text{th},\text{min}}/E \) and \( n \) is the number of intact fibers in the steady state. Using that in the steady state \( \frac{dn}{dt} = 0 \), the number of breaking fibers equals the number of re-bonding fibers.

The number of re-bonding fibers is \( (1 - n)^2 \ast p_{\text{max}} \), while the number of breaking fibers has two parts. The first part contains those fibers that haven’t survived the first sintering step. The second part contains the fibers that survived the sintering process but reached their breaking threshold. For the uniform distribution, when we reach the steady state, in every subset \([\varepsilon, \varepsilon + \Delta \varepsilon]\) of the density function the actual load on the fibers is also uniformly distributed between 0 and \( \varepsilon \). If we order these subsets such that we put the “youngest” fiber on the top, than the bottom-most fibers will break in the next step. Using this, we can write down a quadratic equation and by solving it we get the steady state value for the fraction of intact fibers \( n \).

\[
n = B - \sqrt{B^2 - 1} \hspace{1cm} (17)
\]

\[
B = 1 + \frac{\dot{\varepsilon}_g \varepsilon_{\text{max}}}{p_{\text{max}} \left( \varepsilon_{\text{max}}^2 - \varepsilon_{\text{th},\text{min}}^2 + 2 \dot{\varepsilon}_g \varepsilon_{\text{th},\text{min}} \right)} \hspace{1cm} (18)
\]

This analytic formula is in perfect agreement with the simulations (see the dashed horizontal lines of Fig. 5 left). Since now we know the exact form of the probability density function, we also know exactly the fraction of fibers in each subset \([\varepsilon, \varepsilon + \Delta \varepsilon]\). Since the actual load on the fibers in a subset is uniformly distributed, it is very simple to define the average load acting on a fiber in a certain subset, and in this way we can give a closed analytic form for the asymptotic load bearing capacity of the system:

\[
\sigma_g = \frac{\int f(\varepsilon) \left[ \varepsilon_{\text{max}}^3 - \varepsilon_{\text{th},\text{min}}^3 \right] - \frac{(1 - n)^2 p_{\text{max}}}{4 \varepsilon_{\text{max}}^2} \left[ \varepsilon_{\text{max}}^2 - \varepsilon_{\text{th},\text{min}}^2 \right]}{6} \hspace{1cm} (19)
\]

The predictions of this equation can be seen on the right side of Fig. 5 as dashed horizontal lines.
V. CONCLUSIONS

We have presented a semi-analytic solution for the recently proposed model of I. Reiweger et al. [4]. The obtained fraction of intact fibers, and the constitutive curve of the system are in a very good agreement with the simulation results. After a certain time the system reaches a steady state. We were able to calculate the asymptotic value of the $n_i(t)$ and $\sigma_g(t)$ functions in the simplest case. In the experiments the constitutive curve developed a decreasing tail after a plastic plateau. Our model is not able to predict this behavior. By introducing a sintering probability that decreases with time (ageing), this behavior might be also obtained. The model can also be modified by giving a finite limit to the re-bonding possibility of the fibers. This would be similar to the work of F. Raischel et al. [7].

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FIG. 1: The fat horizontal lines represent the rigid snow slabs. The vertical thinner lines are the fibers, representing the weak layer. The dashed line was the original position of the left-most fiber, before displacement $\Delta x$.

FIG. 2: The behavior of $\sigma_{th, \text{min}}$, see Eq. (4) (left) and Eq. (5) (right). The sintering time is $T_s = 21$. 
FIG. 3: The left figure shows the number of intact fibers as a function of global deformation, while on the right we see the constitutive behavior of the system. The solid black lines are predicted by the semi-analytic formula of Eq. (9) (left) and Eq. (15) (right), while the symbols are simulation results.

FIG. 4: The figure shows how the density distribution function of the strength threshold changes due to the effect of sintering. The (red) circles show the original density function of the uniform distribution, while the (green) triangles show the steady state distribution. $T$ is the time needed for the system to reach the dynamic equilibrium.
FIG. 5: This figure shows the fraction of intact fibers (left) and the constitutive behavior (right) of the system. The symbols represent the simulation results, the solid lines are the predictions of the semi-analytic formula, and horizontal dashed lines are showing the predicted values for the steady state.