Subcritical crack growth: the microscopic origin of Paris's law

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We investigate the origin of Paris's law which states that the velocity of a crack at subcritical load grows like a power law, $da/dt \sim (\Delta K)^{\beta}$, where ΔK is the stress intensity factor amplitude. Starting from a damage accumulation function proportional to $(\Delta\sigma)^{\gamma}$, $\Delta\sigma$ being the stress amplitude, we show analytically that the asymptotic exponent β can be expressed as a piecewise-linear function of the damage accumulation exponent γ, namely, $\beta = 6 - 2\gamma$ for $\gamma < \gamma_c$, and $\beta = \gamma$ for $\gamma \geq \gamma_c$. In this way, here we discover the existence of a critical value $\gamma_c = 2$ characterized by a scaling law with a critical exponent separating two regimes of different linear functions $\beta(\gamma)$. We performed numerical simulations to confirm this result for finite sizes. Finally, we also study the introduction of bounded disorder in the breaking thresholds and find that below γ_c disorder is relevant, i.e., the exponent β is changed, while above γ_c disorder is irrelevant.

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In 1963 Paris and Erdogan [1] postulated that under fatigue loading a subcritical crack grows with a velocity that increases with the stress intensity factor, or equivalently the crack length, as a power law with an empirically determined exponent m. Numerous experiments showed that this law is valid over at least three orders of magnitude for a very wide spectrum of materials [2–5]. The Paris law had huge implications in engineering since it allowed to predict the residual lifetime of loaded materials quantitatively. Today this law constitutes part of basic knowledge and is taught in elementary courses on mechanics [6, 7]. Although there has been attempts to derive the Paris law in terms of geometrical and crack-tip damage-accumulation models (see e.g. Ref. [6] and references therein), no work has been capable of establishing a firm connection between the Paris exponent and microscopic parameters. It is the aim of our paper to present an analytical and a numerical calculation which relates the Paris exponent m to the local damage accumulation law. This constitutes a micro-macro derivation of the Paris law.

For bodies under cyclic load with stress amplitude $\Delta\sigma$, subcritical fatigue crack growth is governed by the Paris law [1],

$$\frac{da}{dt} \sim (\Delta K)^{\beta} = C\left(\Delta\sigma\sqrt{a}\right)^{\beta}, \tag{1}$$

where a is the crack half-length, $\Delta K \sim \Delta\sigma\sqrt{a}$ is the stress-intensity factor range, and β ($\equiv m$) is a material-dependent exponent. Integration over time leads to the Basquin law [8],

$$t_f \sim (\Delta\sigma)^{-\beta}, \tag{2}$$

which relates the lifetime t_f (or equivalently the number of loading cycles to failure) to the stress amplitude. Let us note that while the Basquin law applies for high-cycle fatigue with an exponent β depending on the material structure, at low-cycle fatigue the corresponding empirical relation, $t_f \sim (\Delta\epsilon)^{-\lambda}$, where $\Delta\epsilon$ is the plastic deformation, is called Coffin-Manson law and has an exponent λ that is remarkably close to 2, at least for polycrystalline single-phased metals [9, 10].

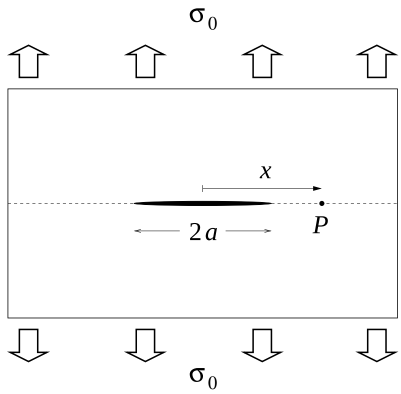

Figure 1: A very thin crack of length $2a$, propagating along the dashed line in a two-dimensional medium subject to a stress σ_0 at infinity. Point P is at a distance x from the crack center.

Recent work [11] has shown that, in the rupture of fiber-bundle models subject to fatigue damage governed by damage-accumulation functions of the form $(\Delta\sigma)^{\gamma}$, the Basquin law is verified with an exponent given by the damage-accumulation exponent γ. This brings the question as to whether it is possible to establish a direct connection between a microscopic damage accumulation form and the Paris law, via the corresponding exponents.

In order to address this question, we consider a linear crack of length $2a$, in a two-dimensional medium subject to a transverse external stress σ_0 exerted very far from the crack, as depicted in Fig. 1. We model the medium as composed of small elements connected by stiff elastic springs, with separation δr. In the continuum limit, and within linear elasticity theory, the transverse stress at a point along the crack line, a distance x from the midpoint of the crack, is given by [12]

$$\sigma\left(x;a\right)=\sigma_0\frac{x}{\sqrt{x-a}\sqrt{x+a}}. \tag{3}$$

Close to the crack tip ($x\simeq a$), the stress diverges as

$$\sigma\left(x;a\right)=\frac{K}{\sqrt{2\pi\left(x-a\right)}}, \tag{4}$$

defining the stress intensity factor $K=\sigma_0\sqrt{\pi a}$.

We assume that the medium is under cyclic load, with an external stress varying between $\sigma_{\min}$ and $\sigma_{\max}$, leading to a stress-intensity-factor range $\Delta K=\left(\sigma_{\max}-\sigma_{\min}\right)\sqrt{\pi a}$. We further assume that fatigue damage is the sole factor driving crack growth, which happens only along the crack line. Specifically, the half-length of the crack increases by δr when the accumulated damage at the crack tip reaches a threshold value F_{thr}. Damage increments are assumed to be given by

$$\delta F\left(x;a\right)=f_0\delta t\left(a\right)\left[\Delta\sigma\left(x;a\right)\right]^{\gamma}, \tag{5}$$

where $\delta t\left(a\right)$ is the number of cycles during which the crack has length $2a$, f_0 is a constant related to the time scale, and $\Delta\sigma\left(x;a\right)$ is calculated from Eq. (3) with σ_0 varying between $\sigma_{\min}$ and $\sigma_{\max}$. The power-law dependence of the damage increment can be justified by invoking concepts of self-similarity [13], and can be seen as a stress-amplification exponent [14].

Since the accumulated damage at point x when the crack has length $2a$ is given by

$$F\left(x;a\right)=F\left(x;a-\delta r\right)+\delta F\left(x;a\right), \tag{6}$$

it follows that

$$\delta t\left(a\right)=\frac{F_{\mathrm{thr}}-F\left(a+\delta r;a-\delta r\right)}{f_0\left[\Delta\sigma\left(a+\delta r;a\right)\right]^{\gamma}}. \tag{7}$$

Here, in order to avoid the appearance of infinities, we assume that the spring attached to an element at position x experiences a stress given by $\sigma\left(x+\delta r;a\right)$. To a first approximation, this is consistent with the fact that linear elasticity theory must break down in the immediate vicinity of the crack tip, giving rise to a fracture process zone or plastic zone [6, 15].

Numerical iteration of the above equations reveals a time dependence of the crack length $2a$ which, for large values of a, reproduces the Paris-Erdogan law, Eq. (1), with a γ-dependent exponent β, as shown in the inset in Fig. 2. Notice that β (as determined from the slopes of the log-log plots) seems to reach a minimum value for $\gamma\simeq 2$. This is confirmed by a systematic calculation of β as a function of γ, obtained by power-law fits of da/dt

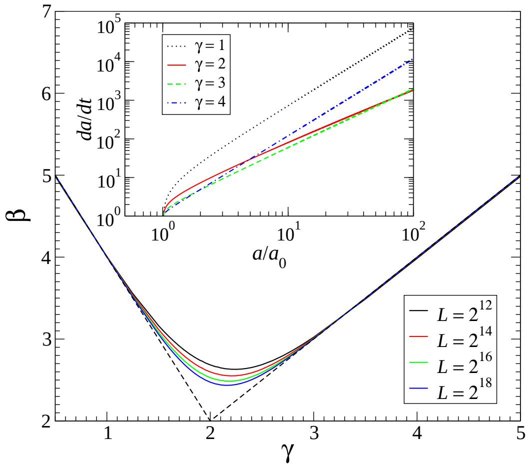

Figure 2: (Color online) Inset: Log-log plot of the time derivative of the crack half-length a, as a function of a/a_0, for various values of the damage exponent γ. In each case, the time scale was adjusted so that da/dt approaches unity as a approaches the initial value $a_0=100\delta r$. Main panel: Dependence of the Paris-Erdogan exponent β on the damage exponent γ, for different values of the maximum crack length L. The dashed curve corresponds to Eq. (8).

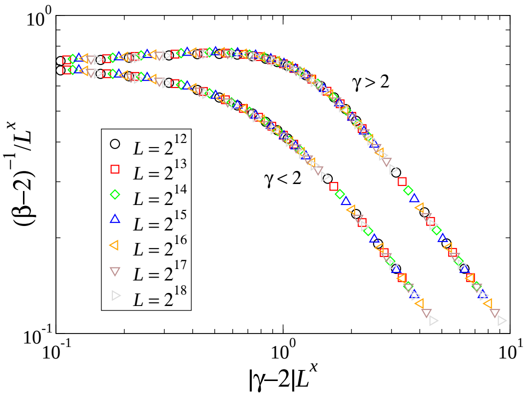

Figure 3: (Color online) Scaling plot of β versus γ for different values of the maximum crack length L, showing conformance with Eq. (9). The best data collapse is obtained with $x=0.089$.

versus a, using various system sizes L. The fits were extracted from points whose abscissa values lie between 90% and 100% of L. As shown in the main panel in Fig. 2, there are strong finite-size effects around $\gamma_c=2$. As $L\to\infty$, the minimum in each curve shifts slowly towards $\beta=2$ at $\gamma_c=2$, suggesting an asymptotic form

$$\beta=\begin{cases}6-2\gamma, & \text{for } \gamma<\gamma_c\\ \gamma, & \text{for } \gamma>\gamma_c\end{cases}. \tag{8}$$

This can be checked by a finite-size scaling hypothesis,

$$\frac{1}{\beta\left(L\right)-2}=\begin{cases}L^x\mathcal{F}_-\left(\left|\gamma-2\right|L^x\right), & \text{for } \gamma<\gamma_c\\ L^x\mathcal{F}_+\left(\left|\gamma-2\right|L^x\right), & \text{for } \gamma>\gamma_c\end{cases}, \tag{9}$$

which, as shown in Fig. 3, is nicely fulfilled by our numerical data with $x = 0.089$. Both scaling functions behave as $\mathcal{F}_{\pm}(u) \sim u^{-1}$, for $u \gg 1$, in agreement with the suggestion that, in the continuum limit ($L \gg \delta r$), β should be a linear function of γ for both $\gamma < \gamma_c$ and $\gamma > \gamma_c$. We have therefore found evidence that $\gamma = \gamma_c$ is a critical point with a critical exponent x.

Linear dependence of Eq. (8) is also revealed by an analytical treatment of Eqs. (5)-(7), which we now discuss. From the time $\delta t\,(a_0)$ during which the crack has length $2a_0$, we can calculate the instantaneous damage $F\,(a_0 + 2\delta r; a_0)$ at the next crack tip position $a_0 + 2\delta r$, and then recursively determine the damage fraction at the successive crack tip positions, $G_n \equiv F\,(a_0 + (n+1)\delta r; a_0 + (n-1)\delta r)\,/F_{\mathrm{thr}}$, using $\delta t\,(a_0 + (n-1)\delta r)$ and $F\,(a_0 + (n+1)\delta r; a_0 + (n-2)\delta r)$. As a result, it is possible to write the recurrence relation

$$G_n + \sum_{k=1}^{n} g_{n,k} G_{k-1} = \sum_{k=1}^{n} g_{n,k}, \tag{10}$$

with $G_0 = 0$ and

$$g_{n,k} = \left[\frac{\Delta\sigma\,(a_0 + (n+1)\delta r; a_0 + (k-1)\delta r)}{\Delta\sigma\,(a_0 + k\delta r; a_0 + (k-1)\delta r)}\right]^{\gamma}. \tag{11}$$

Notice that $g_{n,k}$ is determined by the ratio between the stress amplitudes at position $n\delta r$ and at the crack tip, when the crack has grown by a distance $2k\delta r$. From Eqs. (7) and (3), it follows that, for $a = a_0 + n\delta r$, with $n \gg 1$,

$$\frac{da}{dt} = \frac{\delta r}{\delta t(a)} \sim \frac{n^{\frac{\gamma}{2}}}{1 - G_n}, \tag{12}$$

so that the scaling behavior of the crack growth rate depends on the scaling behavior of G_n. This can be investigated by looking at Eqs. (10) and (11), from which, if $\delta r \ll a_0 \ll n\delta r$, we obtain the asymptotic forms

$$g_{n,k} \simeq \begin{cases} (2\delta r/a_0)^{\frac{1}{2}\gamma}, & \text{for } k \ll a_0/\delta r; \\ 2^{\frac{\gamma}{2}} k^{-\frac{1}{2}\gamma}, & \text{for } a_0/\delta r \ll k \ll n; \\ (n - k + 2)^{-\frac{1}{2}\gamma}, & \text{for } k \lesssim n. \end{cases} \tag{13}$$

It is also easy to show that $g_{n,k}$ has a single minimum at $k_{\min} \simeq n/\sqrt{3}$, so that, as $n \to \infty$, the sum on the right-hand side of Eq. (10) has a power-law divergence $n^{1-\frac{1}{2}\gamma}$ for $\gamma < 2$, while it converges to a finite value $C_\gamma = \zeta(\frac{1}{2}\gamma) - 1$ for $\gamma > 2$, where $\zeta(x)$ is the Riemann zeta function. It follows that G_n approaches $C_\gamma/(1 + C_\gamma) < 1$ for $\gamma > 2$, and from Eq. (12) we immediately see that $\beta = \gamma$. On the other hand, numerical calculations show that, for $\gamma < 2$, G_n asymptotically approaches unity; although we were not able to derive an analytic expression, it can be readily checked numerically that in this case

$$1 - G_n \sim n^{-3\left(1 - \frac{1}{2}\gamma\right)}, \tag{14}$$

leading, when combined with Eq. (12), to $\beta = 6 - 2\gamma$.

For $\gamma > \gamma_c = 2$, in the light of Eq. (2), the prediction $\beta = \gamma$ is compatible with the result obtained in Ref. [11], stating that the Basquin-law exponent is given by the damage exponent γ. This is no longer valid for $\gamma < \gamma_c$. However, the models studied in Ref. [11] involve randomness in both fatigue and stress thresholds as additional ingredients.

Thus, in order to investigate the effects of disorder on the Paris exponent, we introduce a distribution of values of the fatigue thresholds with lower (upper) cutoff F_1 (F_2), so that each lattice point has a local threshold $F_1 \leq F_{\mathrm{thr}}(x) \leq F_2$. An obvious consequence of the disorder is that, depending on the disorder strength and on the damage exponent γ, it is possible that a point far from the crack tip reaches its local fatigue threshold *before* a point closer to the crack tip. This leads to the occurrence of rupture avalanches, in much the same way as in fiber-bundle models (see e.g. Ref. [16]).

Some predictions on the conditions for the appearance of avalanches can be drawn from the analytical approach presented above. Avalanches will occur if, for some crack length $2a$, the local threshold at position $a + \delta r$ is less than the accumulated damage at that position when the crack had length $2(a - \delta r)$,

$$F_{\mathrm{thr}}\,(a + \delta r) < F\,(a + \delta r; a - \delta r)\,. \tag{15}$$

As the fatigue thresholds are no longer the same for all points, Eq. (10) ceases to be valid. However, $F\,(a + \delta r; a - \delta r)$ can still be written as a linear combination of the thresholds of the points closer to the crack center, with coefficients related to the stress ratios $g_{n,k}$. In the absence of any previous avalanches, and for $\gamma > \gamma_c = 2$, the value of $F\,(a + \delta r; a - \delta r)$ is limited by F_2 times the asymptotic value of G_n,

$$F\,(a + \delta r; a - \delta r) \leq F_2 \frac{C_\gamma}{1 + C_\gamma}. \tag{16}$$

Since $F_1 \leq F_{\mathrm{thr}}\,(a + \delta r)$, we conclude that avalanches will occur if

$$F_1 - F_2 \frac{C_\gamma}{1 + C_\gamma} \leq 0 \quad \Rightarrow \quad \frac{F_1}{F_2} \leq \frac{C_\gamma}{1 + C_\gamma}. \tag{17}$$

On the other hand, for $\gamma < \gamma_c$, this argument indicates that any finite amount of disorder leads to the occurrence of avalanches.

Confirmation of these predictions, as well as further information on the effects of disorder, can be obtained from numerical calculations. In order to analyze the results of those calculations, it is useful to integrate the Paris-Erdogan law to obtain

$$1 - \left(\frac{a_0}{a}\right)^{\frac{1}{2}\beta - 1} = Bt, \tag{18}$$

where B is a β-dependent constant related to the inverse rupture time. If we introduce a uniform distribution of

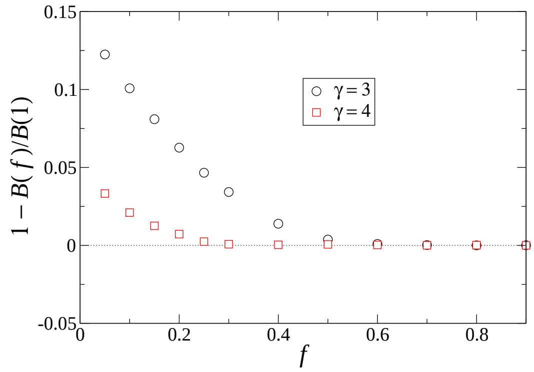

Figure 4: (Color online) Plots of the deviation from unity of the ratio $B(f)/B(1)$ between the coefficient of Eq. (18) in the presence and in the absence of disorder, as a function of the ratio $f = F_1/F_2$ between the lower and upper distribution cutoffs.

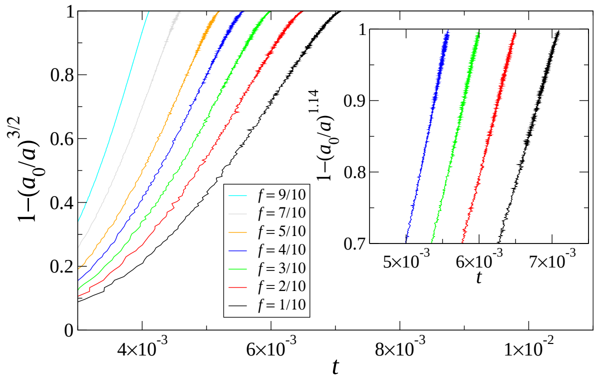

Figure 5: (Color online) Rescaled plots of the crack length for $\gamma = \frac{1}{2}$ and various ratios $f = F_1/F_2$ between the minimum and maximum values of the fatigue thresholds. Main panel: rescaling as given by Eq. (18). Inset: rescaling with an exponent $1.14 \neq \frac{1}{2}\beta - 1$. All curves were obtained for system size $L = 2^{15}$ and averaged over 500 disorder realizations.

fatigue thresholds,

$$P(F_{\rm thr}) = \frac{1}{F_2 - F_1}\theta(F_2 - F_{\rm thr})\,\theta(F_{\rm thr} - F_1), \qquad (19)$$

numerical iteration of Eqs. (3) and (5) — modified by the introduction of local thresholds — shows that, for $\gamma > \gamma_c$, plots of $1-(a_0/a)^{\frac{1}{2}\beta-1}$ for large times (not shown) remain straight lines, with the same value of $\beta = \gamma$ as in the absence of disorder; however, for sufficiently small values of $f = F_1/F_2$, with a fixed average threshold $(F_1+F_2)/2$, the coefficient B becomes disorder-dependent, as shown for $\gamma = 3$ and $\gamma = 4$ in Fig. 4. The values f_c of f at which the plots start to deviate from the horizontal dotted line are compatible with the predictions of Eq. (17), namely $f_c \simeq 0.62$ and $f_c \simeq 0.39$ for $\gamma = 3$ and $\gamma = 4$, respectively. Indeed, we find $1 - B(f)/B(1) \sim (f_c - f)^\alpha$ for $f < f_c$, with the exponent α around 2.5, but we cannot exclude that the exponent might depend on γ.

For $\gamma < \gamma_c$, we confirm the occurrence of avalanches for any amount of disorder. Also, as shown in Fig. 5 (main panel), the prediction of Eq. (18) is no longer verified, indicating that the Paris-Erdogan exponent β is actually modified by the introduction of disorder. The new exponent β', which recovers the linear behavior predicted by Eq. (18), is almost independent of the ratio f, for sufficiently strong disorder, as also shown in Fig. 5 (inset), and we have checked that in this limit the ratio $(\beta' - 2)/(\beta - 2)$ is independent of γ, being given approximately by 0.76.

This distinction between the effects of the disorder for $\gamma < \gamma_c$ and $\gamma > \gamma_c$ is reminiscent of the Harris criterion for the relevance of disorder on the critical behavior of ferromagnetic models [17], according to which disorder changes the critical exponents of the system if the uniform specific heat exponent α is positive, but leaves them unchanged if α is negative.

We have been able to derive analytically and confirm numerically the Paris law and found that its exponent β is a function of the damage exponent γ describing the microscopic damage accumulation law. To our big surprise we discovered that $\gamma_c = 2$ is a critical point characterized by a scaling law and a critical exponent which separates two regimes with different linear functions $\beta(\gamma)$. In addition we also studied the role of disorder and found again that $\gamma_c = 2$ plays a special role: disorder is relevant below it and irrelevant above it.

Our results can have far-reaching consequences in the understanding and control of subcritical crack propagation. On one hand the discovered relation between the damage and the Paris exponents, which in principle could be checked experimentally, could help to predict lifetimes of samples by studying the velocity of small cracks. As we found, the value $\gamma_c = 2$ is very special. For example, to avoid the influence of disorder one must try to stay above γ_c. Finally, it is important to mention that the exponent γ of the damage accumulation law is material dependent and could therefore play a central role in the engineering design to increase the robustness and optimize the mechanical performance of the system.

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