Existence and uniqueness of the solution to the multi-body contact status problem

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The multi-body contact status problem is the question of how to choose the status (sliding or non-sliding) of critical contacts in a quasi-static assembly of bodies interacting via Coulomb friction. Critical contacts occur when the ratio of tangential to normal contact force attains its maximum allowed value. This problem is difficult because there are $2^{M_c}$ possible ways to choose the contact status, where the number $M_c$ of critical contacts can easily be in the hundreds. We consider the special case where the bodies are disks, and the hardness of the bodies is small compared to the confining pressure, so that the deformation of the disks is small. We show that if one requires that the contact status be consistent with the relative motion at each critical contact, there is one and only one way to choose the contact statuses.

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I. INTRODUCTION

Soils are often modeled microscopically by assemblies of particles. It is hoped that the study of these models will lead to a better understanding of the large-scale behavior of granular media. Numerical implementations of these models are now widely used: one need only browse through a recent conference proceeding [1] to realize the wide variety of applications these methods have found.

The theoretical study of these models is much less developed, due their inherent multi-body nature. Progress has been made in two different cases. About twenty years ago, it was proposed that the kinetic theory of dense gases could be adapted to rapid granular flows [2]. This method provided a systematic way to develop continuum equations describing granular “gases”. Recently, progress has been made for very slow deformations of particle assemblies subjected to a confining pressure, which are often called “granular solids” [3–9]. The very small displacements of the particles enables one to consider them as being nearly fixed in place. The granular packing is therefore considered as a a force-transmitting network. Results of this approach include a deeper understanding of isostaticity of packings of frictionless particles [3, 5, 6], the microscopic origins of strain [3–5], and the stability of packings [8, 9]. A related numerical method [4, 5, 7, 13] has also been developed, but not widely applied.

One difficulty in applying these methods to frictional particles is the problem of choosing the contact status. When two frictional particles touch, the contact between them can be either sliding or non-sliding. If the contact status are specified, the motion of each particle in response to a changing external force can be calculated. But then the chosen contact status must be consistent with the calculated motion. For example, the force at sliding contacts must always oppose the relative motion of the two touching particles. If a particular choice of contact status leads to consistency at all contacts, that choice is called globally consistent.

In a numerical method, one must find the globally consistent choice of status through an iterative procedure: one first guesses the status of each contact, calculates the resulting velocities, and then checks for consistency. If an inconsistency is found, the statuses are modified, and then this new choice must be checked for consistency. It is important to realize that the number of ways that the contact status can be chosen is very large: if there are $M_c$ contacts whose status must be determined, then there are $2^{M_c}$ ways to do so, and values of $M_c > 100$ are common.

From a theoretical perspective, one can ask about the existence and uniqueness of the globally consistent choice of status. Perhaps there are packings where there is no way to choose the statuses in a consistent way. What would that mean? On the other hand, given the large number of ways, one might suspect that there are many different globally consistent ways to choose the statuses. Then one would like to know what process chooses among these possibilities in an experiment. We show however, that if the packing is stable, then the globally consistent state always exists and is unique.

This paper is organized as follows. Sec. II presents an overview of the paper, detailing the questions posed in the introduction, and sketching the results of the rest of the paper. Readers not wishing to savor the details of the proof may read this section, and then skip directly to Sec. VII for a discussion of the results. The stiffness matrix for frictional disks is derived in Sec. III, and a mathematical formulation of the premises of the proof are given in Sec. IV. The proof for small numbers of contacts is given in Sec. V, and for an arbitrary number in Sec. VI.

II. SYNOPSIS

A. Coulomb friction and contact status

In this paper, we will deal with an assembly of disks, interacting via Coulomb friction and subjected to a slowly changing force. As a concrete example, consider a bi-axial box, where a granular sample composed of disks is
The contacts between the disks are assumed to be cohesionless and have Coulomb friction. This means that the following conditions are obeyed:

\[ F_n \geq 0, \quad \bar{F} \equiv \mu F_n - |F_t| \geq 0. \]  

(1)

Here, \( F_n \) and \( F_t \) are the normal and tangential components of the contact force. The constant \( \mu \) is the Coulomb friction coefficient. The set of forces \((F_n, F_t)\) that obey these conditions is called the Coulomb cone because they form a cone when plotted in the \((F_n, F_t)\) plane. The Coulomb cone is shown in Fig. 2 as a shaded region. Each contact in a packing has a normal force \( F_n \) and a tangential force \( F_t \) and can thus be represented by a point in the \((F_n, F_t)\) plane. Eq. (1) requires that all such points fall within the Coulomb cone. Contacts whose forces lie on the boundary of the Coulomb cone \(|F_t| = \mu F_n\) are called critical contacts, and they play a crucial role in the evolution of the packing.

The contact forces are generated by deformations of the particles. Thus, when two touching particles move relative to one another, the contact forces change. For contacts in the interior of the Coulomb cone, there is a law relating the relative motion of the contacts to the change in the contact forces. Contacts following this law are called non-sliding, because the particles’ surfaces remain locked together. This law cannot always be applied to critical contacts, for they may leave the Coulomb cone. Thus critical contacts can follow another law that guarantees that they remain on the boundary of the Coulomb cone. Such contacts are called sliding, because the particles’ surfaces slide relative to one another. Which law that the contacts follow — sliding or non-sliding — is called the contact status.

The question now arises: how does one choose the status of the critical contacts? Should they be sliding or non-sliding? The usual approach is to make the contact sliding only when necessary, i.e. whenever making it non-sliding would cause it to leave the Coulomb cone.

In this paper, we formulate the question slightly differently, and ask if a particular choice of status is consistent with a given velocity. This is consistency is determined by the principle of minimum sliding: The status ‘sliding’ is consistent if, and only if, making the particle non-sliding would cause it to leave the Coulomb cone. The two possible statuses are mutually exclusive: if ‘sliding’ is consistent, then ‘non-sliding’ is inconsistent, and vice versa.

In the following, we are not only concerned about single contacts, but entire packings with many contacts whose status must be chosen. We will refer to a way of choosing the statuses of these contacts as the global contact status or simply the state of the packing. If a global contact status leads to consistency at all contacts, the packing is said to be in the globally consistent state. This paper is concerned with the existence and uniqueness of the globally consistent state.

### B. The Stiffness matrix

In Sec. III, we show that if the movement is quasi-static and the particles are quasi-rigid, the behavior of the packing is piece-wise linear. Thus time can be divided into intervals \([t_i, t_{i+1}]\) during which the velocities of the particles are linearly related to the change in forces:

\[ \frac{df_{\text{ext}}}{dt} = hv, \]

(2)

where \( f_{\text{ext}} \) represents the external forces \((F_x\) and \(F_y\) for the biaxial box), \( v \) is the velocities of the particles, and \( h \) is called the stiffness matrix.

The problem is only piece-wise linear because the stiffness matrix \( h \) depends on the contact status. Whenever
a contact status changes, therefore, $h$ must be modified. Therefore, the times $\{t_i\}$ which define the intervals of linearity are the times when a contact moves from the interior of the Coulomb cone and reaches the boundary, triggering the change of contact status at one or more contacts.

A possible trajectory of a contact in the $(F_n, F_i)$ plane is shown in Fig. 2. The contact moves in a straight line from $A$ towards $B$ since Eq. (2) is linear. When it reaches the point $B$, it becomes critical, and thus sliding, because otherwise it would leave the Coulomb cone. It thus moves along the cone until point $C$, where it enters again back into the interior of the cone. This re-entry can only occur if a second contact (whose trajectory is not shown in Fig. 2) becomes critical and sliding, thus modifying the stiffness matrix in such a way that the relative velocities at the first contact are inconsistent with its status. Thus the first contact is obliged to become non-sliding and enter back into the Coulomb cone.

With this example we are now ready to formulate the two questions answered by this paper.

C. The question of existence and uniqueness

The first question answered by this paper concerns the existence of a globally consistent choice of status. As pointed out above, a change in status of a single contact modifies the stiffness matrix, and thus the relative velocities at all contacts, including the one changing its status. Therefore, one cannot fix the status by inspecting the velocities. Instead, the iterative procedure described in the introduction must be used. But is this procedure guaranteed to terminate? Is it possible that both ‘sliding’ and ‘non-sliding’ would be inconsistent with their corresponding velocities? For example, when the contact reaches $B$ in Fig. 2, it is clear that the non-sliding status is inconsistent, for this would mean continuing the trajectory outside the Coulomb cone. So the contact must become sliding. This status change modifies all velocities within the packing. Could it happen that the new velocities at the concerned contact are inconsistent with its new status? In this case, a consistent choice of status would not exist.

We might console ourselves that there is probably a globally consistent choice of status, because there is such a large number of possibilities. If there are $M_c$ critical contacts, there are $2^{M_c}$ possible ways to assign their status. In large simulations with thousands of particles, there can easily be hundreds of sliding contacts, meaning that the ways one can assign statuses cannot even be numbered by a computer using 64-bit integers.

This might make us worry less about existence, but now we must worry about uniqueness. With so many possibilities, perhaps there are many globally consistent ways to assign status. This would mean that the stiffness matrix, combined with the principle of minimum sliding, would not be a complete description of the system. Some physical process must decide between the different possibilities. This unknown process would have been left out of the model, leading to indeterminacy in the same way that neglecting particle deformations leads to force indeterminacy. We would then have to ask what that physical process could be. One possibility is sound waves. As we show below, the quasi-static assumption amounts to removing ‘fast’ processes like sound waves. When a contact changes status, there is probably a “negotiation” between the critical contacts, mediated by sound waves, that establishes their status. In the quasi-static limit, this period of negotiation becomes a single point in time, and it is assumed that the velocity-status consistency rule suffices to determine the new status. Non-uniqueness of the choice of contact status means that the details of this negotiation must be taken into account.

D. Proof of uniqueness and existence

The central finding of this paper is that the globally consistent state always exists and is unique.

We now sketch the proof. There are two premises. First, we assume that all possible choices of contact status lead to a stable packing. The packing is stable if

$$v^T h v > 0,$$

for a certain (large) class of relevant vectors $v$. The second premise is the observation that the left hand side of Eq. (2) is not modified by the status of the contacts. Thus if we consider two different choices $X$ and $Y$ for the global contact status, one has

$$\frac{dF_{ext}}{dt} = h^X v^X = h^Y v^Y,$$

where $h^X$ is the stiffness matrix obtained if one chooses $X$, and $h^Y$ is obtained by choosing $Y$. The corresponding velocities are $v^X$ and $v^Y$.

From Eqs. (3) and (4) it is possible to derive a series of inequalities, from which one may deduce the following theorem: if the status of any set of contacts changes, the consistency of at least one of those contacts must also change. This statement is sufficient to prove both existence and uniqueness.

To investigate the consequences of this statement, let us cast it into a different form. A particular choice of status corresponds to a $M_c$-bit binary number, $S \in \{0, 1\}^{M_c}$, with each bit corresponding to the status of a single contact. For concreteness, let us say $S_\alpha = 1$ if contact $\alpha$ is sliding, and $S_\alpha = 0$ if it is non-sliding. To check the consistency of a given choice $S$ of contact status, we would construct the corresponding stiffness matrix $h$, solve Eq. (2) for $v$, and then check for consistency at each critical contact. The result of this procedure can be represented by a second $M_c$-bit binary number $C = C(S)$, where each bit gives the consistency of a critical contact, i.e. $C_\alpha = 1$ if contact $\alpha$ is consistent, and $C_\alpha = 0$
otherwise. Now, the theorem can be stated as follows: Changing any number of bits of $S$ causes at least one of the corresponding bits in $C$ to change. This means that no two different values of $S$ can lead to the same value of $C$. There are $2^{M_C}$ possible choices for $S$, and $2^{M_C}$ possible values for $C$, so each possible value of $C$ must be associated with a unique value of $S$. This applies also to $C = 111\ldots1$, corresponding to global consistency.

This result can be elegantly stated using a more mathematical language. The process of determining the consistency of a choice of contacts defines a mapping $C$ of one $M_C$-bit binary number to another:

$$C : \{0, 1\}^{M_C} \rightarrow \{0, 1\}^{M_C}$$

(5)

The central result of the paper is simply: the mapping $C$ is bijective. This proves the existence and uniqueness of the consistent choice since $C$ is a mapping from $\{0, 1\}^{M_C}$ onto itself.

### III. THE STIFFNESS MATRIX

#### A. Particle interaction model

We suppose that the grains interact through cohesionless repulsion and Coulomb friction. When two grains first touch, two springs are created, one in the tangential and the other in the normal direction. The springs obey Hooke’s law so that the normal and tangential contact forces $F_n$, $F_t$ are proportional to the spring elongations $D_n$, $D_t$. To this restoring force, we add a linear damping to model the dissipation of energy:

$$F_n = -K_n D_n - \Gamma_n V_n, \quad F_t = -K_t D_t - \Gamma_t V_t,$$

(6)

where $K_n$ and $K_t$ are the spring constants, $\Gamma_n$ and $\Gamma_t$ are viscous damping coefficients, and $V_n$ and $V_t$ are the normal and tangential relative velocities. Here, $F_n > 0$ is interpreted as pushing the particles apart, and $D_n < 0$ occurs when the particles overlap. The conditions in Eq. (1) will be enforced by placing conditions on $D_n$ and $D_t$, as explained below.

The springs are stretched by the relative motion of the particles, as long this does not violate any of the conditions in Eq. (1). When the contact is in the interior of the Coulomb cone, any motion is possible, so one has

$$\frac{dD_n}{dt} = V_n, \quad \frac{dD_t}{dt} = V_t,$$

(7)

where $V_n$ and $V_t$ are just the relative velocities at the point of contact:

$$V_n = (\vec{v}_i - \vec{v}_j) \cdot \hat{n} \hat{n},$$

$$V_t = (\vec{v}_i - \vec{v}_j) \cdot \hat{u} + r_i \omega_i + r_j \omega_j,$$

(8)

where $\vec{v}_i$, $\omega_i$, and $r_i$ are the velocity, angular velocity and radius of particle $i$, and $i$ and $j$ label the touching particles. Throughout this paper, capital letters indicate to quantities concerning contacts, and small letters quantities concerning particles.

Now let us consider how to handle sliding contacts. It is helpful to define

$$\vec{V} = \mu \frac{K_n}{K_t} V_n + V_t \text{sgn} D_t,$$

(9)

Note that if the contact is non-sliding,

$$\frac{d\vec{V}}{dt} = -(K_t + \Gamma_t) \vec{V}.$$

(10)

where $\vec{F}$ is defined in Eq. (1). For contacts on the boundary of the Coulomb cone, we have $\vec{F} = 0$. The sign of $\vec{V}$ determines whether such contacts leave or remain within the Coulomb cone when made non-sliding. If $\vec{V} < 0$, the contact will move into the interior of the Coulomb cone ($\vec{F} > 0$). If $\vec{V} > 0$, the point would leave the Coulomb cone. The principle of minimum sliding can thus be restated: The status ‘sliding’ is consistent if, and only if, $\vec{V} > 0$. This formulation is used throughout this paper.

Now let us consider how to treat sliding contacts. We will suppose that Eq. (6) is always valid, but set $\Gamma_t = 0$ and constrain the spring elongations much change in such a way $\vec{F}$ remains 0. This can be accomplished if we use the first equation in Eq. (7) but replace the second with

$$\frac{dD_t}{dt} = -\mu \frac{K_n}{K_t} V_n \text{sgn} D_t.$$

(11)

Once the contact forces are known, the net force $\vec{f}$ and torque $\tau$ on each particle can be computed:

$$\vec{f} = \sum_{\alpha} F_{\alpha,n} \hat{n}_\alpha + F_{\alpha,t} \hat{t}_\alpha,$$

$$\tau = r \sum_{\alpha} F_{\alpha,t},$$

(12)

where the sums are taken over all the contacts that the concerned particle makes with its neighbors, and $r$ is its radius.

#### B. Matrix formulation

It is useful to consider the proceeding equations in matrix form. To do so, we must gather the various quantities into vectors. To begin with, we can group the force and torque exerted on particle $i$ into a vector $\vec{f}_i$, and the contact forces exerted by a contact $\alpha$ into a vector $F_{\alpha}$:

$$\begin{pmatrix} f_{i,x} \\ f_{i,y} \\ \tau_i/r_i \end{pmatrix}, \quad F_{\alpha} = \begin{pmatrix} F_{\alpha,n} \\ F_{\alpha,t} \end{pmatrix}.$$

(13)

It is often convenient to group these vectors together into high-dimensional quantities concerning all the particles.
or contacts in the packing:

$$f = \begin{pmatrix} f_{1,x} \\ f_{1,y} \\ \vdots \\ f_{N,x} \\ f_{N,y} \end{pmatrix} = \begin{pmatrix} t_1/r_1 \\ \vdots \\ t_N/r_N \end{pmatrix},$$

(14)

and

$$F = \begin{pmatrix} F_{1,x} \\ F_{1,y} \\ \vdots \\ F_{M,x} \\ F_{M,y} \end{pmatrix} = \begin{pmatrix} F_{1,1} \\ F_{1,2} \\ \vdots \\ F_{M,1} \\ F_{M,2} \end{pmatrix}.$$

(15)

Here $N$ is the number of bodies whose motion must be considered, and $M$ is the number of contacts between these bodies. Note that $f$ is a member of the $3N$-dimensional linear space $\mathbb{R}^{3N}$ that we will call the *particle space*, and that $F \in \mathbb{R}^{2M}$, that we name the *contact space*. In these equations, and throughout this paper, boldface vectors will denote quantities concerning all contacts or particles (i.e. vectors in contact or particle space), whereas underscores indicate quantities associated with a single particle or contact.

Eq. (12) can now be written

$$f_i = \sum_{\alpha=1}^{M} \mathcal{E}_\alpha F_\alpha,$$

(16)

where $\mathcal{E}_\alpha$ is a $3 \times 2$ matrix

$$\mathcal{E}_\alpha = \begin{pmatrix} \chi_{\alpha} h_{ax} & \chi_{\alpha} t_{ax} \\ \chi_{\alpha} h_{ay} & \chi_{\alpha} t_{ay} \\ 0 & |\chi_{\alpha}| \end{pmatrix}.$$  

(17)

This gives the contribution of contact $\alpha$ to the force exerted on particle $i$. The symbol $\chi_{\alpha}$ is defined as

$$\chi_{\alpha} = \begin{cases} 1 & \text{if particle } i \text{ is first in contact } \alpha, \\ -1 & \text{if particle } i \text{ is second in contact } \alpha, \\ 0 & \text{if particle } i \text{ does not participate in contact } \alpha. \end{cases}$$

(18)

For each contact between two grains, one element of $\chi$ is 1, and another is $-1$. $\chi$ is also called the incidence matrix.

Eq. (16) holds for each particle ($i = 1 \ldots N$). All of these equations can be written compactly using the definitions in Eqs. (14) and (15):

$$f = c F.$$

(19)

The $3N \times 2M$ matrix $c$ can be constructed assembling an $N \times M$ array of the $\mathcal{E}_\alpha$. The matrix $c$ maps a vector from contact space into particle space.

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On can consider Eq. (19) as an equation for the unknown contact forces $F$. However, one almost always has $2M > 3N$, meaning that $c$ has at least $2M - 3N$ linearly independent null eigenvectors. Therefore, Eq. (19) does not have a unique solution. This is the force indeterminacy problem discussed elsewhere in the literature [8, 10–12, 14].

Eq. (6) can be written

$$E_i = -K_i D_{ii} - \sum_\alpha V_\alpha, \quad \text{or } F = -KD - \Gamma V,$$

(20)

where

$$K_i = \begin{pmatrix} K_n & 0 \\ 0 & K_t \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} \Gamma_n & 0 \\ 0 & \Gamma_t \end{pmatrix},$$

(21)

and $K$ and $\Gamma$ are $2M \times 2M$ diagonal matrices containing the $K_\alpha$ or the $\Gamma_\alpha$ on the diagonal.

Eq. (8) can be written as

$$V_\alpha = \sum_{i=1}^{N} (\mathcal{E}_\alpha)^T \mathcal{E}_i, \quad \text{or } V = c^T v,$$

(22)

where $c^T$ is the transpose of $c$ [3]. The matrix $c^T$ maps a vector from particle space into contact space. Since the dimension of contact space is larger than that of the particle space, Eq. (22) places restrictions on $V$. Not every vector $V \in \mathbb{R}^{2M}$ is allowed, but only those vectors in the range of $c^T$. Physically, this means that not every relative motion is possible, but only those that can be generated by moving and rotating the particles. The dimension of the range of $c^T$ is at most $3N$. There are thus $2M - 3N$ dimensions in $\mathbb{R}^{2M}$ that are inaccessible. These $2M - 3N$ dimensions are precisely the null space of $c$ [3].

Finally, the relation between $D$ and $V$ in Eqs. (7) and (11) requires careful treatment, due to the different possible contact statuses. Let $\mathcal{S}$ be the set of sliding contacts. We define a $2 \times 2$ matrix $\mathcal{S}_\alpha$ that depends on the status of contact $\alpha$. If $\alpha \notin \mathcal{S}$, $\mathcal{S}_\alpha = 1$, if $\alpha \in \mathcal{S}$:

$$\mathcal{S}_\alpha = \begin{pmatrix} -\mu \frac{h_{at}}{t_{at}} & 0 \\ 0 & \mu \frac{h_{at}}{t_{at}} \end{pmatrix}.$$ (23)

Now the relation between $v$ and $D$ can be written:

$$\frac{dD}{dt} = \mathcal{S}_\alpha V_\alpha, \quad \text{or } \frac{dD}{dt} = S(\mathcal{S}) V.$$ (24)

$S(\mathcal{S})$ is a block diagonal matrix, with the $\mathcal{S}_\alpha$ on the diagonal. It is a function of $\mathcal{S}$, as indicated.

Note that $\mu \frac{h_{at}}{t_{at}}$ in Eq. (23) is a constant. In order for $\mu \frac{h_{at}}{t_{at}}$ to change, the contact must cross the $F_{th}$ axis in Fig. 2. This can only happen if the contact passes through the interior of the Coulomb cone. In that case, the contact would be non-sliding, and Eq. (23) would not be applied. The exception to this occurs when a contact approaches the origin. We are then concerned with a contact that opens or closes, which we are not treating in this paper.
C. Equations of motion

We suppose an external force $f_{\text{ext}}$ is exerted on the particles. The equation of motion of the particles is thus

$$m \frac{dv}{dt} = f + f_{\text{ext}}.$$  \hspace{1cm} (25)

Here, $m$ is a diagonal matrix containing the masses and momenta of inertia of all the grains. We could also write

$$m_i \frac{dv}{dt} = f_i + f_{\text{ext},i}.$$  \hspace{1cm} (26)

with

$$m_i = \begin{pmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & I_i/r_i^2 \end{pmatrix},$$  \hspace{1cm} (27)

where $m_i$ is the mass of particle $i$ and $I_i$ is its moment of inertia.

Combining Eq. (19), (20) and (25) gives

$$m \frac{dv}{dt} = -cKD + f_{\text{ext}}.$$  \hspace{1cm} (28)

This equation can be differentiated once with respect to time, and Eqs. (22) and (24) can be used to obtain

$$m \frac{d^2v}{dt^2} = -cKS \mathbf{v}^T \mathbf{v} \mathbf{c} \mathbf{T} + \frac{dK}{dt} \mathbf{v} + \frac{dc}{dt} \mathbf{v}^T \mathbf{T} c \mathbf{v} + \frac{df_{\text{ext}}}{dt}.$$  \hspace{1cm} (29)

The combination $cKS \mathbf{v}^T \mathbf{v}$ appears often, so we define $h = cKS \mathbf{v}^T$ and write

$$m \frac{d^2v}{dt^2} = -hv - \frac{dc}{dt} \mathbf{v}^T \mathbf{T} c \mathbf{v} + \frac{df_{\text{ext}}}{dt}.$$  \hspace{1cm} (30)

This equation gives the full motion, without approximation, of the disks. Such an equation is solved numerically in the “molecular dynamics” simulation method. On the left hand side is the mass times the acceleration (differentiated by time), and on the right hand side are the forces exerted on the particles (also differentiated by time).

D. Quasi-static balance

If one makes the quasi-rigid and quasi-static assump- tions, then two terms dominate in Eq. (30). The quasi-rigid assumption means that the hardness of the particles is assumed to be much greater than the confining pressure. The quasi-static assumption is that the external forces $f_{\text{ext}}$ change much slowly than any timescale associated with the contact forces. We describe the consequences of each of these assumptions below.

Let us begin with the quasi-rigid assumption. We will compare the first two terms on the right hand side. From Eq. (17), we see that $\frac{dv}{dt}$ will be proportional to $\frac{d\mathbf{v}}{dt}$. Carrying out this differentiation:

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left( \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right) = \frac{v_i - v_j}{|\mathbf{x}_i - \mathbf{x}_j|} \cdot \mathbf{t} \sim O \left( \frac{V}{R} \right).$$  \hspace{1cm} (31)

Here, $R$ is a typical particle radius, and $V$ a typical relative velocity. Now we can estimate the sizes of the first two terms on the right hand side of Eq. (30):

$$hv = cKS \mathbf{v}^T \mathbf{v} \mathbf{c} \mathbf{T} \sim O \left( KV \frac{D}{R} \right),$$  \hspace{1cm} (32)

where we have used $c \sim S \sim O(1)$. The quasi-rigid assumption implies that the spring lengths $D$ are much smaller than a typical particle radius $R$, or $D/R \ll 1$. Thus the second term on the right hand side is much smaller than the first. There are some situations, however, when the second term can again become important, such as cyclic loading [15].

Now let us proceed to the quasi-static assumption. To express the separation of time scales, we write

$$\frac{d}{dt} = \frac{d}{dt_0} + \epsilon \frac{d}{dt_1},$$  \hspace{1cm} (33)

where the variable $t_0$ measures the fast time scale, and $t_1$ the long time scale. The presence of $\epsilon \ll 1$ before the derivative with respect to $t_1$ means that $t_1$ measures slow movements, and thus differentiation results in small quantities.

The assumption of quasi-staticity can be expressed by saying that the particle positions $\mathbf{x}$ depend on both $t_0$ and $t_1$, but the external force $f_{\text{ext}}$ depends only on the slow time $t_1$:

$$\mathbf{x} = \mathbf{x}_0(t_0) + \mathbf{x}_1(t_1),$$  \hspace{1cm} (34)

and the $O(1)$ terms of Eq. (30) are

$$m \frac{d^2\mathbf{v}_0}{dt^2} = -h \mathbf{v}_0 + c \mathbf{v}_1.$$  \hspace{1cm} (35)

After differentiation by time, we have

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1.$$  \hspace{1cm} (36)

This equation resembles that of a damped, harmonic oscillator, with two differences. First, it is an equation for the velocity, not the position. And second, it is a vector equation, not a scalar one. Nevertheless, it has the same properties as a damped, harmonic oscillator. The matrices $m$ and $c \mathbf{v}^T \mathbf{T} c$ are positive definite, as $m$ and $\mathbf{T}$ are both diagonal matrices with positive entries. The stiffness matrix $h$, however, is not necessarily positive definite. If there are vectors $\mathbf{v} \neq 0$ such that $\mathbf{v}^T \mathbf{T} \mathbf{v} < 0$, then $h$ acts like a negative number in Eq. (36), and $\mathbf{v}_0$ grows exponentially a the short time scale. This is “motion through an instability” [8]. In this case, one does not obtain the quasi-static balance; rather the packing is unstable and is set in motion.

On the other hand, if $\mathbf{v}^T \mathbf{v} = 0$, nothing catastrophic happens; $\mathbf{v}_0$ decays exponentially to zero, although there will be a net displacement of the grains. If $\mathbf{v}^T \mathbf{v} > 0$, then $\mathbf{v}_0$ oscillates as it decays. In both cases, $\mathbf{v}_0 \to 0$ as
\( t_0 \to \infty \). It then makes sense to consider the \( O(\epsilon) \) terms. There remain simply
\[
\frac{df_{\text{ext}}}{dt} = hv_1.
\] (37)

This is the quasi-static balance.

Eq. (37) also shows what happens when \( h \) has a null eigenvalues, i.e., when \( hv = 0 \) for \( v \neq 0 \). Such vectors \( v \) correspond to motions that do not change the contact forces, and are called “mechanisms” [3]. They have two possible effects. If \( v \) is orthogonal to the left hand side of Eq. (37), nothing happens. An example is if the packing is set in uniform motion, and \( f_{\text{ext}} \) is constructed so as to exert no net force on the grains, as it is in the biaxial test. Another example is a “rattler” - a grain which has no contacts. On the other hand, if \( v \) is not orthogonal to \( df_{\text{ext}} \), then Eq. (37) has no solution. Recall that each null eigenvalue of a matrix reduces the dimension of its range by one. If the range of \( h \) is reduced so that \( df_{\text{ext}} \) no longer belongs to it, then Eq. (37) has no solution. Terms from Eq. (30) must then be included. The most logical choice is the inertia. But this means that static equilibrium will be lost, and disk will be set in motion. This is “motion through a mechanism”.

E. Application to biaxial tests

To give a concrete example, consider the biaxial test sketched in Fig. 1. The forces applied to the walls are
\[
F_x = L_y \left[ p_0 - q(t) \right], \quad F_y = L_x \left[ p_0 + q(t) \right],
\] (38)
where \( p_0 \) is the pressure exerted on the sample, and \( q(t) \) is the deviatoric strain, which varies slowly in time.

We consider the walls as particles, and insert them into the various vectors and matrices. Let us take particle 1 to be the left wall, 2 to be the bottom wall, 3 to be the right wall, and 4 to be the top wall. Then define
\[
1_p = \begin{pmatrix}
(L_y, 0, 0)^T \\
(0, L_x, 0)^T \\
(-L_y, 0, 0)^T \\
(0, -L_x, 0)^T \\
\vdots
\end{pmatrix},
\] (39)
where all elements that are not shown (those that correspond to the grains) vanish. In addition, we define
\[
1_q = \begin{pmatrix}
(-L_y, 0, 0)^T \\
(0, L_x, 0)^T \\
(L_y, 0, 0)^T \\
(0, -L_x, 0)^T \\
\vdots
\end{pmatrix},
\] (40)
Again, all elements not shown vanish. Now \( f_{\text{ext}} \) is simply
\[
f_{\text{ext}} = p_0 1_p + q(t) 1_q,
\] (41)
and Eq. (37) is simply
\[
\left( \frac{dq}{dt} \right) 1_q = hv.
\] (42)

IV. CONDITIONS NEEDED TO SHOW UNIQUENESS

In this section, we present the two assumptions needed to prove the uniqueness and existence of a solution. The first condition is that all possible states be stable. The second condition is that the external forces be independent of the state of the packing.

A. The stability condition

A packing is stable if the quadratic form from
\[
Q(v, S) = v^T h(S)v > 0,
\] (43)
As we saw in Sec. III D, the motion cannot be assumed to be quasi-static when \( Q \leq 0 \).

The quadratic form plays an important role in this paper, so we will discuss how it can be calculated. If we recall the definition \( h = cKSc^2 \), and group factors in a suggestive way, we have
\[
Q(v, S) = [v^T c]^T K S [c^T v] = v^T [K S] v
\] (44)
The matrix \( K S \) is block diagonal, with each block corresponding to a contact. Thus the \( Q(v, S) \) reveals itself to be simply a sum over contacts:
\[
Q(v, S) = \sum_{\alpha=1}^{M} V_{\alpha}^T K_{\alpha} S_{\alpha} V_{\alpha}.
\] (45)
If the contact is non-sliding, then \( S_{\alpha} = 1 \), and the contribution of the contact \( \alpha \) is
\[
Q_{\alpha}^{NS}(v) = K_{\alpha} V_{\alpha}^2 + K_{\alpha} V_{\alpha}^2,
\] (46)
and if the contact is sliding
\[
Q_{\alpha}^{S}(v) = Q_{\alpha}^{NS}(v) - K_{\alpha} V_{\alpha} \tilde{V}_{\alpha} \text{sgn} D_{t, \alpha}.
\] (47)
In the following, it is useful to use Eq. (9) and replace \( V_{\alpha, \alpha} \) with \( \tilde{V}_{\alpha} \). Eq. (47) becomes
\[
Q_{\alpha}^{S}(v) = Q_{\alpha}^{NS}(v) - K_{\alpha} V_{\alpha} \text{sgn} D_{t, \alpha}.
\] (48)
Now let us define
\[
\tilde{F}_{\alpha} = K_{\alpha} V_{\alpha} \text{sgn} D_{t, \alpha},
\] (49)
so that Eq. (48) becomes
\[
Q_{\alpha}^{S}(v) = Q_{\alpha}^{NS}(v) - \tilde{F}_{\alpha} \tilde{V}_{\alpha}.
\] (50)
Therefore, the stability condition is
\[ Q(\mathbf{v}, \mathbb{S}) = Q(\mathbf{v}, \emptyset) - \sum_{\alpha \in \mathbb{S}} \dot{F}_\alpha \dot{V}_\alpha > 0. \] (51)

Note that \( Q(\mathbf{v}, \emptyset) > 0 \), because the contribution of each contact must be positive. This means the only way to obtain an unstable packing is for the sliding contacts to make large and negative contributions to \( Q \).

In the following, it will be necessary to compare \( Q \) for different states. If the sliding contacts present in a given state are divided into two disjoint sets \( \mathbb{S}_1 \) and \( \mathbb{S}_2 \) \((\mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset)\), then
\[ Q(\mathbf{v}, \mathbb{S}_1 \cup \mathbb{S}_2) = Q(\mathbf{v}, \mathbb{S}_1) + \sum_{\alpha \in \mathbb{S}_2} \dot{F}_\alpha \dot{V}_\alpha, \] (52)

### B. Independent load condition

Let us consider two different states \( X \) and \( Y \), each with a different set of sliding contacts. Let \( \mathbb{S} \) be the set of contacts sliding in both states, \( \mathbb{X} \) be the set of sliding contacts unique to \( X \) and \( \mathbb{Y} \) be those unique to \( Y \). Let \( \mathbf{v}^X \) be the velocities in state \( X \) and \( \mathbf{v}^Y \). Similarly, \( \mathbf{h}^X = \mathbf{h}(\mathbb{S} \cap \mathbb{X}) \) and \( \mathbf{h}^Y = \mathbf{h}(\mathbb{S} \cap \mathbb{Y}) \). If the externally applied force is independent of contact status,
\[ \frac{df_{\text{ext}}}{dt} = \mathbf{h}^X \mathbf{v}^X = \mathbf{h}^Y \mathbf{v}^Y, \] (53)
or both \( \mathbf{v}^X \) and \( \mathbf{v}^Y \) are the velocities caused by the same external forces, but with different stiffness matrices.

Eq. (53) can be rewritten
\[ \mathbf{h}^X \mathbf{v}^X - \mathbf{h}^Y \mathbf{v}^Y = \mathbf{cK}[S^X \mathbf{v}^X - S^Y \mathbf{v}^Y] = 0. \] (54)

Now let us multiply this equation from the left by \((\mathbf{v}^X - \mathbf{v}^Y)^T\):
\[ (\mathbf{v}^X - \mathbf{v}^Y)^T \mathbf{K} [S^X \mathbf{v}^X - S^Y \mathbf{v}^Y] = 0. \] (55)

This again is simply a sum over contacts:
\[ \sum_{\alpha = 1}^M (\mathbf{v}_\alpha^X - \mathbf{v}_\alpha^Y)^T \mathbf{K}_\alpha [S^X \mathbf{v}_\alpha^X - S^Y \mathbf{v}_\alpha^Y] = 0. \] (56)

There will be four types of contributions:
- Contacts which slide in \( X \) but not in \( Y \) (the set \( \mathbb{X} \)),
- Contacts which slide in \( Y \) but not in \( X \) (the set \( \mathbb{Y} \)),
- Contacts which are sliding in both \( X \) and \( Y \) (the set \( \mathbb{S} \)),
- Contacts which are non-sliding in both \( X \) and \( Y \).

For the last two classes of contacts, \( S^X = S^Y \), so their contributions here will be the same as to the quadratic form. For contacts \( \alpha \in \mathbb{X} \) the contribution is:
\[ K_\alpha(\mathbf{v}_{\alpha,n}^X - \mathbf{v}_{\alpha,n}^Y)^2 + \mu K_\alpha(\mathbf{v}_{\alpha,n}^X - \mathbf{v}_{\alpha,n}^Y)(\mathbf{v}_{\alpha,n}^X - \mathbf{v}_{\alpha,n}^Y). \] (57)

Defining \( \dot{F}_\alpha^{XY} = \dot{F}_\alpha^X - \dot{F}_\alpha^Y \), this quantity can be rewritten as
\[ Q_\alpha^{(NS)}(\mathbf{v}^X - \mathbf{v}^Y) + \dot{F}_\alpha^{XY} \dot{V}_\alpha^X. \] (58)

In the same way, the contribution of contacts \( \alpha \in \mathbb{Y} \) is
\[ Q_\alpha^{(NS)}(\mathbf{v}^X - \mathbf{v}^Y) + \dot{F}_\alpha^{XY} \dot{V}_\alpha^Y. \] (59)

Thus Eq. (55) becomes
\[ Q(\mathbf{v}^X - \mathbf{v}^Y, \mathbb{S}) = \sum_{\alpha \in \mathbb{X}} \dot{F}_\alpha^{XY} \dot{V}_\alpha^X - \sum_{\alpha \in \mathbb{Y}} \dot{F}_\alpha^{XY} \dot{V}_\alpha^Y. \] (60)

## V. SMALL NUMBERS OF SLIDING CONTACTS

In preparation for treating the general case, we will consider the problem of a packing that may slide at two different contacts \( \beta \) and \( \gamma \). The four different possible states are shown in Fig. 3, and labeled \( A, B, C, \) and \( D \). We will use superscripts to indicate quantities belonging to each state. For example \( \mathbf{v}^A \) are the particle velocities in state \( A \) and \( S^C \) is the status matrix in state \( C \). The system starts in state \( A \) with no sliding contacts. Then contact \( \beta \) reaches the boundary of the Coulomb cone and becomes sliding, and the packing moves to state \( B \). When contact \( \gamma \) moves to the Coulomb cone, the system could move to states \( C \) or \( D \).

![FIG. 3: The states considered in Secs. VA and VB. The states are labeled by A, B, C, and D. Next to each letter is the set of sliding contacts. The packing begins in state A with no sliding contacts. Then contact \( \beta \) reaches the Coulomb cone and becomes sliding, and the packing moves to state B. When contact \( \gamma \) moves to the Coulomb cone, the system could move to states C or D.](image-url)
when contact $\gamma$ reaches the boundary. If both states $C$ and $D$ were consistent, the system could move to either $C$ or $D$, and the solution would not be unique. In this section, however, we will show that the consistent state exists and is unique.

A. One sliding contact

Let us first consider how the contact $\beta$ reaches the boundary of the Coulomb cone. It starts somewhere within the cone, that is with $\tilde{V}_\beta > 0$ [see Eq. (1)]. As the contact moves toward the boundary, $\tilde{F}_\beta$ decreases and then vanishes when $\beta$ reaches the boundary. Therefore, Eq. (10) requires that

$$\tilde{V}_\beta^A > 0.$$  \hspace{1cm} (61)

One usually supposes that $\beta$ begins to slide when it reaches the boundary. The system moves from state $A$ to $B$, and $h^A$ must be replaced with $h^B$. But will the state $B$ be consistent? Recall that the principle of minimum independent load condition to the transition between two states $A$ and $B$. Setting $X = A, Y = B, S = \mathbb{X} = \emptyset$, and $\mathbb{Y} = \{\beta\}$, Eq. (60) becomes

$$Q(v^A - v^B, \emptyset) = -\tilde{F}_\beta^{AB} \tilde{V}_\beta^B.$$  \hspace{1cm} (62)

If state $A$ is stable, the quadratic form must be positive, leading to

$$\tilde{F}_\beta^{AB} \tilde{V}_\beta^B < 0,$$  \hspace{1cm} (63)

which gives us some information about the sign of $\tilde{V}_\beta^B$, but also unfortunately involves the unknown quantity $\tilde{F}_\beta^{AB}$. More information can be obtained by requiring state $B$ to be stable:

$$Q(v^A - v^B, \{\beta\}) > 0,$$  \hspace{1cm} (64)

and after using Eq. (52)

$$Q(v^A - v^B, \emptyset) - \tilde{F}_\beta^{AB} (\tilde{V}_\beta^A - \tilde{V}_\beta^B) > 0,$$  \hspace{1cm} (65)

and finally using Eq. (62):

$$\tilde{F}_\beta^{AB} \tilde{V}_\beta^A < 0.$$  \hspace{1cm} (66)

Together Eqs. (66) and (63) show that $\tilde{V}_\beta^A$ and $\tilde{V}_\beta^B$ have the same sign. We already showed that $\tilde{V}_\beta^A > 0$ in Eq. (61), thus $\tilde{V}_\beta^B > 0$ as well. Therefore state $B$ is compatible, and the solution exists.

Before proceeding, let us pause to note that the reasoning we have just employed does not depend on state $A$ being without sliding contacts. Define $\mathbb{A}$ to be the set of contacts sliding in state $A$. If we simply replace the empty set in Eq. (62) and (65) with $\mathbb{A}$, and $\{\beta\}$ with $\{\beta\} \cap \mathbb{A}$ in Eq. (64), the reasoning remains unchanged. Thus we have a general statement:

First order state change theorem: If two states $A$ and $B$ differ only in the status of a single contact $\beta$, then $\text{sgn} \tilde{V}_\beta^A = \text{sgn} \tilde{V}_\beta^B$.

B. Two sliding contacts

Suppose now that the system is in state $A$, when a second contact $\gamma$ reaches the boundary of the Coulomb condition. The system can now move to either state $C$ where both $\beta$ and $\gamma$ slide, or to $D$, where only $\gamma$ slides. We know that the system cannot return to $A$, because $\tilde{V}_\beta^A > 0$, so that state will no longer be consistent.

The first order state change theorem shows that the solution is unique. Only a single contact is different between states $C$ and $D$, so the theorem states that

$$\text{sgn} \tilde{V}_\beta^C = \text{sgn} \tilde{V}_\beta^D.$$  \hspace{1cm} (67)

If $\text{sgn} \tilde{V}_\gamma^C = \text{sgn} \tilde{V}_\gamma^D = 1$, then contact $\beta$ must slide, and $C$ is consistent but not $D$. On the other hand, if $\text{sgn} \tilde{V}_\gamma^C = \text{sgn} \tilde{V}_\gamma^D = -1$, then contact $\beta$ must be non-sliding, and $D$ is consistent but not $C$.

Now let us check the existence of the solution. Suppose $\text{sgn} \tilde{V}_\gamma^C = \text{sgn} \tilde{V}_\gamma^D = 1$, meaning the system must move to state $C$. In order for this state to be compatible, we must also have $\tilde{V}_\gamma^C > 0$. The first order state change theorem guarantees that

$$\text{sgn} \tilde{V}_\gamma^B = \text{sgn} \tilde{V}_\gamma^C,$$  \hspace{1cm} (68)

$B$ and $C$ differ only by the contact status of $\gamma$. We know that $\tilde{V}_\gamma^B > 0$, since $\gamma$ arrived on the boundary while the system was in state $B$. Thus $\tilde{V}_\gamma^C > 0$ also.

What happens if $\text{sgn} \tilde{V}_\gamma^C = \text{sgn} \tilde{V}_\gamma^D = -1$? In this case, the system must move to state $D$ is this state compatible? The first order state theorem says $\text{sgn} \tilde{V}_\gamma^D = \text{sgn} \tilde{V}_\gamma^A$, but this does not help prove compatibility, since $\tilde{V}_\gamma^A$ has an unknown sign. However, we do know that $\tilde{V}_\gamma^B > 0$, so let us now search for away to relate $\tilde{V}_\gamma^B$ to $\tilde{V}_\gamma^D$. If we apply the independent load condition to states $B$ and $D$, we obtain

$$Q(v^B - v^D, \emptyset) = \tilde{F}_\beta^{BD} \tilde{V}_\beta^B - \tilde{F}_\gamma^{BD} \tilde{V}_\gamma^D.$$  \hspace{1cm} (69)

Requiring all four states $A$, $B$, $C$, and $D$ to be stable leads to the inequalities

$$\tilde{F}_\beta^{BD} \tilde{V}_\beta^B > \tilde{F}_\gamma^{BD} \tilde{V}_\gamma^D,$$  \hspace{1cm} (70)

$$\tilde{F}_\beta^{BD} \tilde{V}_\beta^D > \tilde{F}_\gamma^{BD} \tilde{V}_\gamma^D,$$  \hspace{1cm} (71)

$$\tilde{F}_\beta^{BD} \tilde{V}_\beta^B > \tilde{F}_\gamma^{BD} \tilde{V}_\gamma^B,$$  \hspace{1cm} (72)

$$\tilde{F}_\beta^{BD} \tilde{V}_\beta^D > \tilde{F}_\gamma^{BD} \tilde{V}_\gamma^B.$$  \hspace{1cm} (73)

Now, the question of the consistency of $D$ arises when the system cannot go to $C$, i.e., when $\tilde{V}_\gamma^C < 0$. The first order status change theorem then requires $\tilde{V}_\gamma^D < 0$. On
the other hand, we know that $B$ is consistent, so $\tilde{V}_B > 0$. Thus $\text{sgn} \tilde{V}_B^D \neq \text{sgn} \tilde{V}_B^D$. This means that the left hand side of either Eq. (70) or (71) must be negative. Thus the right hand side of these inequalities is smaller than a negative number:

$$\tilde{F}_A^{BD} \tilde{V}_A > 0.$$  

(74)

Similar reasoning with Eqs. (72) and (73) leads to

$$\tilde{F}_B^{BD} \tilde{V}_B < 0.$$  

(75)

showing that

$$\text{sgn} \tilde{V}_B^D \neq \text{sgn} \tilde{V}_B^D \Rightarrow \text{sgn} \tilde{V}_B = \text{sgn} \tilde{V}_B^D.$$  

(76)

Thus, if state $C$ is inconsistent, then $D$ is consistent. Thus the consistent state always exists.

As before, we can generalize what we have shown here. Eqs. (70) through (73) also lead to

$$\text{sgn} \tilde{V}_A^D \neq \text{sgn} \tilde{V}_A^D \Rightarrow \text{sgn} \tilde{V}_A = \text{sgn} \tilde{V}_A^D.$$  

(77)

These two results can be summarized by saying that the sign of either $\tilde{V}_A$ or $\tilde{V}_B$ will not change when the system moves from $B$ to $D$. Combined with the first order status change theorem, one can also show that $\tilde{V}_B$ or $\tilde{V}_B$ cannot both change sign when one moves from $A$ to $C$. Thus we can state

**Second order state change theorem:** If two states $A$ and $B$ differ only in the status of two contacts $\beta$ and $\gamma$, then $\text{sgn} \tilde{V}_A^\beta = \text{sgn} \tilde{V}_B^\beta$ or $\text{sgn} \tilde{V}_A^\gamma = \text{sgn} \tilde{V}_B^\gamma$.

**VI. MANY SLIDING CONTACTS**

In this section, we prove

**The $n$th order state change theorem:** If two states $A$ and $B$ differ only in the status of $n$ contacts $\beta_1, \ldots, \beta_n$, then for at least one contact $\beta_i$, $1 \leq i \leq N$, we have $\text{sgn} \tilde{V}_A^{\beta_i} = \text{sgn} \tilde{V}_B^{\beta_i}$.

As a consequence of this theorem, the consistency at one or more of the contacts $\beta_1, \ldots, \beta_n$ must also change. This is so, because if $\tilde{V}_A$ does not change sign, while the status does change, then the consistency must change. When $\beta_i$ was consistent in $A$, it will be inconsistent in $B$, and vice versa. Thus, this is the theorem mentioned in Sec. (II D).

As we showed there, it is sufficient to show both uniqueness and existence of the globally consistent state.

Let us now prove the theorem. Consider two different states $A$ and $B$. The contacts in $A$ are sliding in $A$ but not in $B$, and the contacts in $B$ are sliding in $B$ but not in $A$. Let $S$ contain contacts that are sliding in both states. We want to show that

$$\text{sgn} \tilde{V}_A^a = \text{sgn} \tilde{V}_B^a \text{ for at least one } a \in A \cap B.$$  

(78)

We will begin by assuming the contrary:

$$\text{sgn} \tilde{V}_A^a \neq \text{sgn} \tilde{V}_B^a \text{ for all } a \in A \cap B,$$  

(79)

and show that this leads to a contradiction.

The independent load condition Eq. (60) implies

$$Q(\nu^A - \nu^B, S) = \sum_{a \in S} F_{AB}^{\alpha} \tilde{V}_A^a - \sum_{a \in S} F_{AB}^{\alpha} \tilde{V}_B^a.$$  

(80)

Now let us assume stability for a third state $C$. The following contacts shall be sliding in $C$:

- All contacts in $S$, who are sliding in both states $A$ and $B$.
- Some contacts that are sliding in $A$ but not in $B$. Let $A' \subset A$ denote these contacts.
- Some contacts that are sliding in $B$ but not in $A$. Let $B' \subset B$ denote these contacts.

Let $C = A \cup B$ and $C' = A' \cup B'$. Note that $C'$ can be any subset of $C$. The stability condition for state $C$ is

$$Q(\nu^A - \nu^B, S) + \sum_{a \in C} F_{AB}^{\alpha} (\tilde{V}_A^a - \tilde{V}_B^a) > 0,$$  

(81)

Combining this with Eq. (80) yields

$$\sum_{a \in A'} F_{AB}^{\alpha} \tilde{V}_A^a + \sum_{a \in A \setminus A'} F_{AB}^{\alpha} \tilde{V}_A^a - \sum_{a \in B \setminus B'} F_{AB}^{\alpha} \tilde{V}_B^a > 0.$$  

(82)

Note that $A'$ and $B'$ are arbitrary, so there is a large number of such relations.

To write these relations in a more compact form, we define

$$\phi^A_a = \begin{cases} F_{AB}^{\alpha} \tilde{V}_A^a, & \text{for } a \in A, \\ -F_{AB}^{\alpha} \tilde{V}_B^a, & \text{for } a \in B, \end{cases}$$  

(83)

with an analogous definition for $\phi^B_b$. Now the relations Eq. (82) can be written

$$\sum_{a \in C'} \phi^A_a + \sum_{a \in C \setminus C'} \phi^B_a > 0.$$  

(84)

And the hypothesis Eq. (79) becomes

$$\text{sgn} \phi^A_a \neq \text{sgn} \phi^B_a \text{ for all } a \in C,$$  

(85)

Now let $C_i$ be a subset of $C$ that contains exactly $i$ elements, and let $C'_i$ be any subset of $C_i$. We will show

$$\sum_{a \in C'_i} \phi^A_a + \sum_{a \in C \setminus C'_i} \phi^A_a > 0 \text{ for } 1 \leq i \leq n.$$  

(86)

where $n$ is the number of elements in $C$. The case $i = 1$ contradicts the hypothesis Eq. (85), showing that it leads
to an inconsistency, and thus proving the $n$th order state change theorem. We will show Eq. (86) by induction, starting from $i = n$ and proceeding down to $i = 1$.

The case $i = n$ is trivial, since $C_n = C$. In this case, Eqs. (84) and (86) identical.

Now let show that if Eq. (86) holds for $i + 1$, then it holds for $i$ also. Suppose that we want to prove Eq. (86) for a given $C_i$ and $C'_i$ with $i < n$. Choose a contact $\beta$ such that $\beta \notin C_i$, but $\beta \in C$. Now Eq. (86) holds for $C_{i+1} = C_i \cup \{\beta\}$. Next, we make two different choices for $C'_{i+1}$ and apply Eq. (86). First we choose $C'_{i+1} = C_i$ and obtain

$$\sum_{\alpha \in C'_i} \phi^B_\alpha + \sum_{\alpha \in C_i \setminus C'_i} \phi^A_\alpha > -\phi^A_\beta, \quad (87)$$

and next we choose $C'_{i+1} = C'_i \cup \{\beta\}$:

$$\sum_{\alpha \in C'_i} \phi^B_\alpha + \sum_{\alpha \in C_i \setminus C'_i} \phi^A_\alpha > -\phi^B_\beta, \quad (88)$$

Note the parallel between these conditions and Eqs. (70) through (73). By Eq. (85) either $\phi^A_\beta$ or $\phi^B_\beta$ is negative. Therefore, the only way for both of these inequalities to hold is if the sums are positive:

$$\sum_{\alpha \in C'_i} \phi^B_\alpha + \sum_{\alpha \in C_i \setminus C'_i} \phi^A_\alpha > 0 \quad (89)$$

This completes the induction step, and thus the proof.

VII. DISCUSSION AND CONCLUSION

A. How restrictive are the assumptions?

One assumption that seems quite restrictive is the use of the linear force law in deriving the stiffness matrix. This is no restriction, because this paper revolves around the question of what occurs at one point in time, when the system must adjust the status of the contacts. Therefore, one could always linearize the force law around the positions of the particles.

A second assumption that seems quite restrictive is that all possible states must be stable. In practice, this is not too much of a restriction, because the system remains near the most unstable state. Therefore, it is plausible that if one of the states is unstable, then the packing will begin to move, and one will not need to find the globally consistent way to assign contact statuses. The system remains near the most unstable state because the only way to obtain instability is when the are many sliding contacts. Therefore, the most unstable state is probably when all $M_c$ contacts are sliding. In practice, the contacts arrive on the Coulomb cone one by one. Thus when a contact arrives on the Coulomb cone, possibly triggering instability, the system is already in a state where $M_c - 1$ contacts are sliding. This state must be stable, otherwise the packing would have already disintegrated. If this state is stable, probably all other states with fewer sliding contacts are stable. Thus the assumption of stability really only amounts to assuming that the state with $M_c$ contacts is stable.

B. Implications of the result

This work suggests that the stiffness matrix, together with the velocity-status consistency rule form a complete description of quasi-static granular material. Since the globally consistent state always exists and is unique, there is no need to appeal to other processes that have been left out of the model to decide between various possible states. Instead of rather brutally setting the particle displacements to zero, as is done in contact dynamics or the force network ensemble, one should consider taking the “quasi-rigid” limit, where the stiffness of the particles diverges, and the displacements become infinitesimally small, but are not set to zero. Taking this limit leads to the stiffness matrix approach discussed in this paper. This work supports the conjecture that the quasi-rigid limit preserves all the necessary physics needed to describe the quasi-static behavior.

Of course, there remain some open questions. For example, the question of opening and closing contacts has not been dealt with. Another restriction that should be lifted is the limitation to two-dimensional disks.

This work should also encourage the use of numerical methods based on the stiffness matrix. The problem faced by these methods is, of course, to find the globally consistent contact status. This work shows that such a state always exists, and is unique. Thus any way to find the state is acceptable. Furthermore, perhaps it is possible to use the results of this paper to design intelligent strategies for finding the globally consistent state.

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