RESTRICTURING OF FORCE NETWORKS

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Abstract We present a model for the evolution of the force network of a granular medium under uniaxial compression. This model is based on the stiffening and the increase of resistance of parallel force lines under the action of individual restructuring events at the contacts between grains and is essentially an inversion of well-known fiber models for brittle tensile rupture. The model can be solved analytically in the experimentally most relevant case in which increase of stiffness and resistance are proportional to each other. The results for the constitutive behavior and the spectrum of the strength of acoustic signals are in good quantitative agreement with experimental measurements.

Keywords: Granular materials.

1. Introduction

The behavior of granular materials has been extensively studied under various conditions due to their scientific and technological importance. Considerable experimental and theoretical efforts have been devoted to obtain a better understanding of the global behavior of granular media in terms of microscopic phenomena which occur at the level of discrete particles [1–7].

In a granular packing the forces are transferred from grain to grain through their contacts which one can consider as nearly point-like. In this sense the forces go along lines which can branch at a grain generating a force network. These force networks can be experimentally visualized by means of photoelasticity using grains made of photoelastic material, putting them between crossed polarizers and shining light through the setup. When the packing is loaded and a certain grain is stressed, it rotates the optical axis and lights up. In this way the force network becomes visible as a beautiful lightened pattern as figure 1 shows. One can even observe in these photoelastic experiments that while the external stress is increased, more and more force lines appear and that each force line undergoes an erratic transformation before reaching a final state at high enough load in which all the grains light up equally [1].
Figure 1. An image of the force chains in a photoelastic, compressed granular medium as viewed between two crossed circular polarizers. The particles, 3-mm Pyrex spheres are surrounded by an index-matching fluid. The force is exerted by a piston that covers the top surface [1].
Subjecting a confined granular packing to an uniaxial compression for small strains a strong deviation from the linear elastic response can be found implying that the system drastically hardens in this regime [2, 7]. Linear elastic behavior can only be achieved asymptotically at larger deformations when the system gets highly compacted. When the external load is decreased again the system shows an irreversible increase in its effective stiffness. Furthermore, under cyclic loading hysteretic behavior is obtained. Microscopically, inside a compressed granular packing, stresses are transferred by the contact of particles. Under gradual loading conditions the particles get slightly displaced changing their contacts and the local load supported by them. Particles lying between lines of the force network do not support any load and can even be removed from the packing without changing its mechanical properties ("rattlers"). The creation and restructuring of percolating force chains implies relative displacements of particles which can be followed experimentally by recording the acoustic waves emitted [8]. Theoretically, this problem has been mainly studied by means of contact dynamics simulations using spherical or cylindrical particles, and cellular automata [4–7]. Computer simulations also reproduced the generation and evolution of force chains, however, the statistics of microscopic restructuring events and their relation to the macroscopic constitutive behavior remained unclarified.

2. Description of the Experiment

In the experiments Conducted by C. Grosse at the IWB of the University of Stuttgart [8], a cylindrical container made of PMMA was filled with glass beads of 5 mm diameter and water. The cylinder has a thickness of 5 mm and a diameter of 140 mm. An uniaxial compression test was carried out applying monotonically increasing displacements at the top level of the glass beads. Experiments were performed under strain controlled conditions at a fixed strain rate, i.e. moving the piston at a constant speed of some mm/minute. Examples of the nonlinear elastic response of the system can be observed in Fig. 3 where the measured force is presented as a function of relative displacement of the cylinder top.

To obtain information about microscopic processes, the acoustic waves emitted due to sudden relative displacements of particles were monitored. Eight acoustic sensors were placed at the container wall to record the signals emitted during the compression of the beads, as can be seen in Fig. 2. Usually, the acoustic emission signal energy is relatively weak and a proper coupling of the sensors is required. To enhance the data quality in regard to the signal-to-noise ratio the space between the beads was saturated with water. The water pressure was kept constant during the course of the experiments by making holes in the upper side of the cylinder. An eight channel transient recorder
Figure 2. Experimental set up and schematic sketch of the array of force chains used in the model. The eight acoustic emission sensors can be observed on the container [8].

Figure 3. Experimental constitutive behavior for different strain rates between 0.1\textit{mm/min} and 1.0\textit{mm/min}.
was used as an analogue-digital converter to enable the storing of the acoustic emission waveforms which implies a sampling rate of 1 ms or 100 ns.

Typically several hundred signals were recorded during the experiment. The inset of Fig. 4 shows the automatically extracted peak amplitudes of the burst signals versus time. The energy is defined as the integral of the acoustic emission signal amplitude. More details of acoustic emission data analysis and especially signal-based techniques can be found in [9–12]. The statistics of restructuring events is characterized by the distribution $D(s)$ of the height $s$ of peaks, which is presented in Fig. 4 on a double logarithmic plot. It can be seen that $D(s)$ shows a power law behavior over two orders of magnitude, the exponent of the fitted straight line is $\delta = 1.15 \pm 0.05$. The data in Fig. 4 are obtained from the eight recorders so that the event size distribution presented is an average over the event size distributions detected by each recorder independently. In this way the influence of the position of the recorders is reduced.
3. Model

We propose a model for the hardening of the individual force lines during compression by an inversion of the fiber models used in rupture mechanics to describe the failure of fiber-reinforced composites [13]. Under applied external load the constitutive behavior of the fiber bundle models are linear for small stresses. With increasing stress, the weakest elements reach their breaking threshold values and the fiber becomes softer. Our model for hardening force networks inverts this situation. The individual lines of the network are considered as fibers which instead of rupturing under tension do harden under pressure due to contact rearrangements.

Fiber bundles are composed of parallel fibers of identical elastic properties but stochastically distributed breaking thresholds. A fiber fails during the loading process when the local load on it exceeds its breaking threshold. Fiber failures are followed by a redistribution of load on the remaining intact fibers according to the range of interaction in the system. The so-called Continuous Damage Model (CDM) [14, 15], is particularly suited to model granular materials since it captures gradual stiffness changes of elements of the model. In our model of compressed packings, force lines formed by particles are represented by an array of lines organized in a square lattice as illustrated in the inset of Fig. 2. A randomly distributed rearrangement threshold \( d \) is assigned to each line of the array from a cumulative probability distribution \( P_0(\frac{d}{d_c}) \), where \( d_c \) denotes the characteristic strength of a force line.

During the compression process, when the local load on a line exceeds its threshold value \( d \) the line undergoes a sudden restructuring as a result of which it becomes stiffer and more resistant. The lines’ stiffness increases in a multiplicative manner, \( i.e. \) the stiffness is multiplied by a factor \( a > 1 \) at each restructuring so that the constitutive equation of a single line after suffering \( k \) restructurings reads as \( \sigma = E_0a^k \sigma_0 \). Here \( E_0 \) denotes the stiffness characterizing single particle contacts, and the exponent \( \alpha \) takes into account possible non-linearities of a single contact like for Hertz law \( \alpha = 1.5 \). After each rearrangement the force chain gets a new threshold value (annealed disorder) from a distribution of the same functional form, but the characteristic strength \( d_c \) of the distribution is increased in a multiplicative way so that after \( k \) rearrangement events the disorder distribution takes the form

\[
P_k(\frac{d}{d_c}) = P_0(\frac{d}{d_0a^k}). \tag{1}
\]

The corresponding distribution density is

\[
p_k(\varepsilon) = \frac{dP_k(\varepsilon)}{d\varepsilon}. \tag{2}
\]
Figure 5. Constitutive behavior of the model for different values of $k_{\text{max}}$ and $\tau$. The cumulative Weibull distribution with $\rho = 2$ and $d_e = 1$ was used [16].
The maximum value of possible restructurings $k_{max}$ is proportional to the number of contacts, and therefore, to $\frac{h}{l_0}$ (see inset of Fig. 2). The ratio $\tau = \frac{q}{q'}$ is a very important parameter of the model, it decides whether the force chain becomes more fragile ($\tau > 1$) or more ductile ($\tau < 1$) as a result of restructur-

Following the derivation of the constitutive behavior of fiber bundles [14, 15] the relation between the stress $\sigma$ and the strain $\varepsilon$ is given by

$$\sigma(\varepsilon) = \sum_{i=0}^{k_{max}-1} a^i E_0 \varepsilon^a [1 - P_i(a^i \varepsilon)] \prod_{j=0}^{i-1} P_j(a^j \varepsilon) + a_{k_{max}} E_0 \varepsilon^a \prod_{i=0}^{k_{max}-1} P_i(a^i \varepsilon).$$

(3)

The first part of Eq. (3) contains the elements which have undergone $i < k_{max}$ restructurings characterized by the local stiffness $E_0 a^i$. The second part includes the elements which have already reached $k_{max}$ and the local stiffness $E_0 a^{k_{max}}$.

When the particles are elastic $a_{k_{max}}$ is a finite value related to the stiffness of an individual particle as $E_p = a_{k_{max}} E_0$. In this cases after all the fibers reach the allowed restructuring event number $k_{max}$ the system will have the same constitutive behavior as a single contact.

The nonlinear stress-strain curves observed in the figures 5 a), b) and c) are in good qualitative agreement with previous experimental and numerical works [2, 4, 5]. The superposition of the curves for $\tau \leq 1$ and big values of $k_{max}$ supports that the solution for $k_{max} = \infty$ will have the same shape. Moreover for $\tau \geq 1$ the shape of the curves suggests the existence of a finite and non-zero critical value of strain $\varepsilon_c$.

For the case $\tau = 1$ the stress and the threshold stress increase with exactly the same pre-factor. Therefore the cumulative distribution for restructuring the $k$th time is independent on $k$ and consequently the same for all the fibers and only depends on the strain value $P_k(a^k \varepsilon) = P_0(\varepsilon)$. For that case the constitutive law Eq. (3) takes the form

$$\sigma(\varepsilon) = E_0 \varepsilon^a (1 - P_0(\varepsilon)) \sum_{i=0}^{k_{max}-1} a^i P_0(\varepsilon)^i + a_{k_{max}} E_0 \varepsilon^a P_0(\varepsilon) .$$

(4)

It can be seen from Eq. (4) that if the maximum number $k_{max}$ of possible restructuring events goes to infinity the stress $\sigma$ has finite values only for $a P_0(\varepsilon) < 1$. In this case the summation can be performed in the first term, while the second term tends to zero, and the constitutive equation takes the form

$$\sigma(\varepsilon) = E_0 \varepsilon^a [1 - P_0(\varepsilon)] \frac{1}{1 - a P_0(\varepsilon)} .$$

(5)
It follows that the stress $\sigma$ diverges when $\varepsilon$ approaches a critical value $\varepsilon_c$, where $\varepsilon_c$ satisfies the equation $P_0(\varepsilon_c) = 1/a$. Expanding $P_0(\varepsilon)$ into a Taylor series at $\varepsilon_c$

$$P_0(\varepsilon) = P_0(\varepsilon_c) + p_0(\varepsilon_c)(\varepsilon_c - \varepsilon) + \ldots,$$

and assuming linear contacts ($\alpha = 1$) the behavior of $\sigma$ in the vicinity of $\varepsilon_c$ reads as:

$$\sigma(\varepsilon) \sim (\varepsilon_c - \varepsilon)^{-1},$$

with $\varepsilon_c = \frac{1}{a}$. This result is also valid for other distributions, while the critical value $\varepsilon_c$ changes. For a cumulative Weibull distribution $P_k(d) = 1 - e^{-\left(\frac{d}{\bar{d}}\right)^\rho}$, a value of critical strain $\varepsilon_c = \left[ \ln \left( \frac{a}{\bar{a} - 1} \right) \right]^{\frac{1}{\rho}}$ was obtained.

It means that the stress $\sigma$ shows a power law divergence when $\varepsilon$ approaches the critical value $\varepsilon_c$. The value of the exponent is universal; it does not depend on the form of disorder distribution $P_0(\varepsilon)$, while the value of $\varepsilon_c$ depends on it.

When the number of force lines is fixed it is possible to obtain analytic results also for the statistics of restructuring events. Under strain controlled loading of a fiber bundle the load on a fiber is determined by its local stiffness and the strain imposed externally. Applying the CDM to compressed granular systems the same assumption is made, i.e. there is no load redistribution among existing force lines, hence, the restructuring of a force line does not affect other elements of the system. Restructuring occurs during the compression process when the local load on a force line exceeds its threshold value. If the new threshold value, assigned to the force line after rearrangement, is smaller than the local load, the force line undergoes successive restructurings until it gets stabilized. The number of steps to reach the new stable state defines the size $s$ of the avalanche of restructuring events.

The number of restructuring events $N_{k,s}$ of size $s$ starting in force chains which have already suffered $k$ restructurings can be deduced as

$$\frac{N_{k,s}(\varepsilon)}{N_o} = p_0(\varepsilon)P_0^{s+k-1}(\varepsilon) \left[ 1 - P_0(\varepsilon) \right],$$

for $s + k \leq k_{\text{max}} - 1$, and

$$\frac{N_{k,s}(\varepsilon)}{N_o} = p_0(\varepsilon)P_0^{k_{\text{max}}-1}(\varepsilon),$$

for $s + k = k_{\text{max}}$

This conditional probability is normalized by the total number of elements $N_o$. The number of events of size $s$, i.e. $D(s)$ is deduced taking into account
the whole compression process and all the starting configurations. Hence, it can be calculated as

\[ D(s) = \sum_{k=0}^{k_{\text{max}}-s} \int_0^{\varepsilon^*} \frac{N_{k,s}(\varepsilon)}{N_o} d\varepsilon + \int_0^{\varepsilon^*} \frac{N_{k_{\text{max}}-s,s}(\varepsilon)}{N_o} d\varepsilon. \]  

(10)

Here \( \varepsilon^* \) denotes the strain value for which all the force chains have reached the maximum number \( k_{\text{max}} \) of reordering events, thus \( P_0(\varepsilon^*) = 1 \) follows. Finally, substituting Eqs. (8,9) into Eq. (10)

\[ D(s) = \sum_{k=0}^{k_{\text{max}}-s} \int_0^1 P^{s+k-1}(1-P) dP + \int_0^1 P^{k_{\text{max}}-1} dP, \]  

(11)

and performing the calculations yields

\[ D(s) = s^{-1}. \]  

(12)

The distribution of microscopic restructuring events exhibits an universal power law behavior with an exponent 1, which is completely independent on the disorder distribution.

The numerical results obtained applying the algorithm of the continuous damage model show how the statistics of the process is governed by the value of \( \tau \). Our results reveal the existence of different local avalanche regimes which was deduced checking the shape of the local avalanche size distributions [16].

The normalized total number of restructuring events \( \frac{N_e}{N_o} \) is presented in Fig. 6 for different system sizes. The collapse of the curves shows that the results are independent on the transversal area of the sample. Moreover, the numerical results suggest the existence of a logarithmic dependence between the total number of events and \( k_{\text{max}} \), in the case \( \tau = 1 \). Integrating the equation (12), \( \frac{N_e}{N_o} \) is deduced as

\[ \frac{N_e}{N_o} = \sum_{s=1}^{k_{\text{max}}} \frac{1}{s} \sim \int_1^{k_{\text{max}}} \frac{ds}{s} = \ln k_{\text{max}}. \]  

(13)

4. Comparison to Experimental Results

Experiments and discrete element simulations [1, 7] have revealed that the number of effective force chains increases during the compression process until it reaches a saturation value. To capture this effect in our model, for the number of elements we prescribe the form \( N(\varepsilon) = N_o G(\varepsilon) \), where \( N_o \) denotes the saturation number of chains, and \( G(\varepsilon) \) has the property \( G(\varepsilon) \rightarrow 1 \) with increasing \( \varepsilon \). Hence, the number of force lines \( dN \) emerging due to an
Figure 6. Total number of restructuring events for different values of $\tau$ and different system sizes.

Figure 7. Constitutive behavior for small strain values measured experimentally and comparison to the theoretical results [8].
Figure 8. Event sizes $s$ occurring during the loading history $\varepsilon$, and their size distribution $D(s)$ on a double logarithmic plot [8].

An infinitesimal deformation increment from $\varepsilon$ to $\varepsilon + d\varepsilon$ is $dN = N_0g(\varepsilon)d\varepsilon$ where $g(\varepsilon) = dG(\varepsilon)/d\varepsilon$. Using Eq. (4) the macroscopic constitutive equation of the compressed granular system for $\tau = 1$ can be cast into the form

$$
\sigma(\varepsilon) = E_o \sum_{i=0}^{k_{\text{max}}-1} \int_0^\varepsilon a_i^i(\varepsilon - \varepsilon_o)^\alpha g(\varepsilon_o)P_0^i(\varepsilon - \varepsilon_o)[1 - P_0(\varepsilon - \varepsilon_o)]d\varepsilon_o \\
+ E_o \int_0^\varepsilon a^{k_{\text{max}}}(\varepsilon - \varepsilon_o)^\alpha g(\varepsilon_o)P_0^{k_{\text{max}}}(\varepsilon - \varepsilon_o)d\varepsilon_o.
$$

The first part of Eq. (13) contains the elements which have undergone $i < k_{\text{max}}$ restructurings characterized by the local stiffness $E_o a^i$. The second part includes the elements which have already reached $k_{\text{max}}$ and the local stiffness $E_o a^{k_{\text{max}}}$. Eq. (13) also takes into account that the local strain of force lines $\varepsilon - \varepsilon_o$ is different from the externally imposed strain value $\varepsilon$ since it also depends on the initial strain $\varepsilon_o$. The integral is performed over the whole loading history to take into account all the generated lines. For explicit calcu-
lations we imposed an exponential form \( N(\varepsilon) = N_o(1 - e^{-\frac{\varepsilon}{\beta}}) \) for the number of chains.

The best fit obtained to the experimental data is presented in Fig. 7 where the force \( F = N_o \sigma \) is plotted against deformation \( \varepsilon \). A power law of an exponent 2.6 was obtained as a good fit to the measured data in a reasonable agreement with former experiments of Ref. [2]. We choose \( N_o = \frac{\Delta \rho}{\Delta \rho_p} = 784 \). An uniform distribution of the restructuring thresholds and a big value of \( k_{max} \) were used. The value of the other parameters are \( E_o = 4600 \frac{N}{m^2}, d_o = 3N, \beta = 0.01 \) and \( a = q = 1.01 \). The value of \( a \) falls close to one meaning that a single restructuring gives rise only to a slight increase of stiffness of a force chain. The small value of \( \beta \) implies that the generation of new force chains stops at a relatively small strain value, and hence, the later rapid increase of \( F \) as a function of \( \varepsilon \) is mainly caused by the hardening of the existing force lines occurring due to restructurings.

Numerical simulations revealed that the universal power law behavior also holds when the gradual creation of force chains is taken into account, i.e. when the system is described by the full Eq. (13). The statistics of restructuring events obtained by Monte Carlo simulations taking also into account the gradual emergence of force lines is presented in Fig. 8. The power law behavior of the analytic prediction is verified. It is important to emphasize that the theoretical results on event statistics (see Fig. 8) are in a very good quantitative agreement with the experimental findings (see Fig. 4).

**Conclusion**

The evolution of effective force chains percolating through a compressed granular system was investigated. The experiments were made by compressing an ensemble of spherical particles in a cylindrical container monitoring the macroscopic constitutive behavior and the acoustic signals emitted by microscopic rearrangements of particles. We have presented a simple model of parallel force lines which harden under load due to restructuring events. If the increase of stiffness and the increase in the restructuring threshold stress are equal, the model can be solved analytically. So that, we applied the continuous damage model of fiber bundles to describe the evolution of the array of force chains during the loading process. The model provides good quantitative agreement with the experimental results. The stress \( \sigma \) shows a power law divergence when \( \varepsilon \) approaches the critical value \( \varepsilon_c \). The value of the exponent is universal; it does not depend on the form of disorder distribution \( P_0(\varepsilon) \), while the value of \( \varepsilon_c \) depends on it. Unfortunately, this divergence in the vicinity of \( \varepsilon_c \) could not be studied with the present experimental setup. The rearrangement of granular materials results in a spontaneous release of acoustic energy radiating waves similar to that observed in other brittle materials under load.
The amplitude distribution of acoustic signals was found experimentally to follow a power law with an exponent $\delta = 1.15 \pm 0.05$ which is in good agreement with the analytic solution of the model $D(s) = s^{-1}$. We argue that this is a consequence of the locality of restructurings due to the absence of load redistribution.

References


