1. – Introduction

Granular materials present a wide spectrum of different basic characteristics and phenomena which are still poorly understood, although well known for a long time. Their unusual behaviour makes them very distinguishable from liquids and solids. A classical example is the formation of a pile of sand when grains are poured onto a table. Unlike water that would spread out on the surface, the final state is a conical heap that grows through sequences of avalanches on the incline, until it reaches a stable configuration whose slope is characterized by a steepest angle, which is called the angle of repose $\theta_r$. This angle is a material property of the granular system [1]. For a sand pile, $\theta_r$ is typically between 25° and 35°, while if the grains are glass beads, it is about 10°–15°, depending mostly on the shape and the surface roughness of the grains, and the humidity of the air.

One of the pioneers of the research on granular media was Osborne Reynolds [2]. He introduced the concept of dilatancy, an effect we encounter while walking on a wet beach: When we press our foot strongly into the sand, a dry halo appears around the foot. Naively one would expect that the pressure of the foot would rather produce a hole in the sand which would fill with water, but the opposite occurs. The physical reason...
for this effect is that the grains were so densely packed that one needs to separate them before deformation occurs.

Individual grains are typically rigid and can have different densities, shapes, sizes and surface roughnesses. Between the grains there can exist interactions characterized by local rotations and point-like contacts, as we are going to see in the following. A consequence of particular interest of the friction between the particles is that granular matter can be set in many stable static configurations. It is due to energy dissipation during the interactions between the grains that sand poured on the floor immediately comes to rest and looses all its kinetic energy. Dissipation introduces a characteristic length scale into a granular system, namely the distance over which the energy decays by a factor $1/e$, since the multiplicative restitution coefficient implies an exponential relaxation [3].

Because of dissipation, every experiment in which a granular system moves must constantly be furnished with energy. This can be done for instance i) by gravity, as in mass flows (e.g. in a hopper, a pipe or an hourglass) [4,5] or in surface flows (like heaps, rotating drums and inclined chutes) [6-8]; ii) by using a loudspeaker or a vibrating conveyor belt [9,10]; iii) for dense systems like packings, energy can be input by shearing the material under gravity for instance in shear cells, which are a very important device to characterize soils [11-13]; iv) the input of energy in granular systems can be done also by means of hydrodynamics (by a surrounding fluid). In the industry, many processes push a fluid through a granular bed (system called a “fluidized bed”) [14-16]; in the hydrology, the formation of river beds on beaches and sedimentation are of major importance; the wind in the desert also acts like a fluid moving the grains and forming beautiful dunes. In fluid sand, dissipation plays an essential role in the evolution of density instabilities, which are responsible for the appearance of clusters with higher concentrations of particles [17,18].

In fact, the density is the most important parameter characterizing the behaviour of a granular system. While loose granular matter can flow like a fluid, dense granular packings behave like a solid. Bagnold was the first to introduce the concept of viscosity $\eta$ and “granular temperature” $T_g$ for flowing sand in 1954 [1], and empirically found the law named after him that $\eta$ is proportional to the shear velocity $\dot{\varepsilon}$ itself, and not a constant as is the case for Newtonian fluids like water or air. In the 1980s, a kinetic gas theory was developed by Savage and Jenkins [19], and Haff [20]. It describes agitated granular media in the framework of fluid dynamics, the only important addition being a dissipation term in the energy flux equations.

Granular materials in densely packed forms are relevant for a wide variety of technical processes. In soil mechanics, as we mentioned above, one of the major problems is the prediction of properties of sheared granular media. Granular geomaterials like sand and silts are good examples of deformation-rate-insensitive plastic solids. The classical description for such deformations, which are mostly irreversible, is given by the so-called non-associate Mohr-Coulomb plasticity theory [21,22].

The non-trivial macroscopic properties observed in granular materials are intimately related to the microscopic nature of the force distributions in the systems, which are
formed through the contacts between the grains. In the next sections, we focus our attention to this topic.

2. – Friction and contact dynamics of grains in densely packed form

Granular matter means a system of particles whose size (or diameter) $d$ is larger than 50 $\mu$m. For $d < 10 \mu$m, thermal agitation is important, and Brownian motion can be seen. For this reason, granular systems are also called systems with zero temperature [23]. Unlike in the atomic regime, where the particles (atoms) interact by overlapping their various orbitals and sharing electrons, the contact between the grains is essentially point-like. Friction between the grains is thus characterized by gigantic local forces. This behaviour leads to important effects, for instance, in the wearing of glass beads, which makes the particles rougher, producing “glass dust”, and thus increasing the specific area of the granular medium [24]. On the other hand, if the particles are made of aluminum, plastic deformation can occur. As a consequence of these interactions, collisions between the particles are typically inelastic, and energy can be dissipated in the form of heat or acoustic emissions.

The friction between two interacting surfaces has been studied for a long time, but still no universally accepted explanation of the basic laws of friction exists. Amonton announced two basic laws of friction in 1699, that were already known by Leonardo da Vinci and are still taught in the present: the first law is that friction is proportional to the normal $F_n$ of the interfaces in contact, i.e. $F_{fr} = \mu F_n$, and the second law is that friction is independent of the area of contact. Later, Coulomb observed that friction was furthermore independent of the relative sliding velocity of the surfaces in contact. However, it has been observed that friction presents features much more complex than this. Bowden and Tabor in 1950 [25] suggested that the existence of friction is related to the roughnesses of the surface areas in contact. This means that the “real” area of contact is in general much smaller than the “apparent” area. Understanding the mechanism of friction in granular systems is a subject of extensive theoretical research [26,27].

The static distribution of stresses in granular materials has been studied for more than one hundred years by researchers in different areas like applied mechanics and geotechnical and chemical engineering. Janssen [28] studied in 1895 the structure of the forces inside a pipe filled with granular material. The schematic model is depicted in fig. 1. We want to know the force distribution of granular matter in a silo. They assumed that the lateral forces (per unit area) exerted by the grains on the wall are proportional to the vertical stresses:

$$F_n = kP,$$

where $P(h)$ is the pressure at the height $h$, and $k$ is a phenomenological coefficient. The friction between the grains and the vertical walls can be written according to Amonton’s law [29]: $F_{fr} = \mu F_n = \mu kP$, where $\mu$ is the related coefficient of friction. Thus, for an
horizontal slice of area $\pi R^2$ and height $\Delta h$, the equilibrium condition reads

$$\pi R^2 \rho g \Delta h = \pi R^2 \frac{dP}{dh} \Delta h + 2\pi R \Delta h \mu k P \implies \rho g = P' + \frac{2\mu k}{R} P,$$

where $\rho$ is the density of the column of granular matter. This expression can be written in a simpler form by introducing the characteristic length $\lambda = R/2\mu k$, which leads to

$$P = \lambda \rho g - \lambda P',$$

which is a first-order differential equation that has as solution pressure profiles of the form

$$P(h) = c \exp \left( -\frac{h}{\lambda} \right) + \lambda \rho g. \quad (2)$$

The value of the constant $c$ can be determined by using the convention that $P(0) = 0$ at the top of the silo, from which it follows that $c = -\lambda \rho g$, and the final equation reads as

$$P(h) = \lambda \rho g \left( 1 - e^{-h/\lambda} \right). \quad (3)$$

Equation (3) has a very important consequence: As we can see from fig. 2, for a pipe filled with grains, the pressure at the base of the container does not rise indefinitely, but saturates on a maximum value $c = \lambda \rho g$, independent of the height. Note that this is very different from the case of a normal fluid, where $P(h) = \rho gh$. The physical origin of this behaviour lies in the way that forces are transmitted in granular media, from grain to grain at their contact points. In a packing of grains, this gives rise to a complex network of force lines that can branch at the contacts, and stress is transmitted laterally. Thus, a normal stress is exerted by the grains on the walls of the container, which support all the extra weight of the material.

On industrial scale, the pipe of fig. 1 would correspond to a silo. Real silos, however, present a funnel on the bottom, leading to a small orifice of diameter $D$ through which the granular material flows out with an essentially constant velocity, as in the case of an hourglass. If $D$ is too small the silo will clog due to the formation of so-called arches. A schematic picture of a single arch is shown in fig. 3 as a simple geometrical model, which is classical in structural mechanics particularly used to describe the static of bridges. The equilibrium condition is the result of the interplay between gravity $g$ and the contacts with the walls of the container. The forces transmitted among the grains of mass $m$ and diameter $d = 2R$ can be decomposed in components $F_x$ and $F_z$. We now determine the form of the arch $z(x)$. Since all the force added to the arch along $x$ goes in the $x$-direction, $F_x$ equals the normal force that reaches the walls and is constant along $x$. The force transmission through the grains in the direction of the vector $d\tilde{s}$ as shown in fig. 3 follows
the equation \( d(\partial \vec{F}/\partial s) = m\vec{g} \) [30]. In the case of an isolated arch (or bridge), illustrated in fig. 3(a), we have

\[
\frac{dF_x}{ds} = 0; \quad \frac{dF_z}{ds} = \frac{mg}{d},
\]

from which it follows that \( F_x = \text{const} \), and \( F_z = (mgs/d) + \text{const} \), or simply \( F_z = mgs/d \), since \( F_z(0) = 0 \), where the origin is at the top of the arch. Thus, we can write

\[
\frac{dx}{dz} = \frac{F_x}{F_z} = \frac{k}{s}, \quad \text{where} \quad k = \frac{F_x d}{mg} = \text{const}.
\]
Fig. 3. - Schematic diagram of an arch clogging the outlet of a silo. In (a) we consider a single arch, and in (b) we take into account the pressure $P$ exerted by the material above it.

But $ds^2 = dx^2 + dz^2 = dz^2(1 + (k^2/s^2))$. Therefore we can write

$$x = \int_0^x \frac{dx}{dz}dz = \int_0^s \frac{k}{s \sqrt{1+(k^2/s^2)}}ds = k \int_0^x \frac{du}{u \sqrt{1+(1/u^2)}} = k \int_0^x \frac{du}{\sqrt{1+u^2}} = k \sinh^{-1} \frac{s}{k},$$

where we did the variable change $u = s/k$. Inverting the equation above, it follows that

$$\sinh \frac{x}{k} = \frac{dz}{dx} \Rightarrow z = k \cosh \frac{x}{k}. \quad (4)$$

We now have to take into account the interaction between the many arches present in the material. Each arch has to support a pressure $P$ resulting from the weight of the material on the top, as shown in fig. 3(b). For simplicity, we assume that the pressure is applied uniformly along the arch, such that the vertical force added at each interval $dx$ is in fact $dF_z = P \, dx$. Since the force that acts on the arch does not vary along the $x$-direction, we now replace the parametrization $s$ by integrating over the variable $x$. Thus, we have at position $x$

$$z = \int_0^x \frac{dz}{dx} dx = \int_0^x \frac{x}{k} dx \Rightarrow z = \frac{2}{k} x^2, \quad (5)$$

which is a parabolic shape. The clogging of silos is an enormous industrial problem and many costly accidents have occurred.

The force distribution in granular materials is a subject of intense theoretical and experimental research [31, 32]. Great effort has been devoted in the last years to the understanding of the global behaviour of granular media in terms of microscopic phenomena which occur at the level of discrete particles [33-39]. The force network inside a granular packing is very heterogeneous, with certain chains along which the stresses are particularly intense. Particles ("rattlers") lying between lines ("cages") of the force network do not support any load, and can even be removed from the packing without changing its mechanical properties. These structures of force lines can be clearly visualized by means
Fig. 4. – Force chains in a granular material as viewed between two crossed-circular polarizers. The particles are 3 mm Pyrex spheres surrounded by a mixture of water and glycerol that matches the index of refraction of the Pyrex [33].

of a simple experimental setup constituted of grains made of photoelastic material, immersed in a liquid mixture between two crossed polarizers. When the packing is loaded, the grains that are stressed rotate their optical axis and light up, forming a beautiful pattern, as shown in fig. 4. It is observed that, as the external stress is increased, more and more force lines appear. Indeed, each force line undergoes an erratic transformation before reaching a stationary state when the load is high enough, regime in which all the grains light up equally [33].

The distribution of stresses in a granular medium as a network of force chains is associated with the non-trivial behaviour of the normal pressures on the floor below a heap of sand [23, 40]. Along the principal axis, there is a “channeling” of forces that percolate through the material in such a manner that two peaks of loads on the bottom appear. This distribution will be different if the sand pile is for instance compressed from the top, since the actual force network will be modified and new force lines can appear. During the compression of a granular material, the creation and restructuring of percolating force chains are associated with relative displacements of the grains, which rearrange their positions until a new stable configuration of the system is achieved. In the next section, we present a model for the evolution of these percolating force chains, that very well captures the restructuring processes at the level of the particles.
3. – Evolution of percolating force chains in compressed granular media

To investigate the evolution of the force chains percolating through a granular system submitted to a uniaxial compression, experiments have been performed in collaboration with the group of Dr. Christian Grosse, at the Institute of Construction Materials (IWB) of the University of Stuttgart. In the experiments, the granular material consisted of glass beads of 5 mm diameter, which were immersed in water inside a cylindrical container made of PMMA with a thickness of 5 mm and a diameter of 140 mm (fig. 5). By moving the traverse at a constant speed of some mm/minute, a uniaxial compression at the top level of the glass beads was carried out under strain controlled conditions. During the compression, the acoustic waves emitted due to sudden relative displacements of the beads were recorded by eight sensors placed at the container wall, as shown in fig. 5. An eight channel (two with 10 MHz and six with 1 MHz) transient recorder was used as an analogue-digital converter to enable the storing of the acoustic emission waveforms and a signal-based data (with a sampling rate of 1 ms or 100 ns). Typically several hundred signals were recorded during the experiment. By averaging over the event size distributions detected by each recorder independently, the influence of the position of the recorders was reduced. The inset of fig. 6 shows a typical averaged time series of the automatically extracted peak amplitudes of the burst signals. The main plot of this figure shows the distribution $D(s)$ of the events with size $s$. The straight line represents a power law fit with exponent $\delta = 1.15 \pm 0.05$, which is followed by the experimental data over two orders of magnitude.

A model for the hardening of the individual force lines during compression has been
recently proposed [41], based on an analogy to fiber models, which are used in rupture mechanics to describe the failure of fiber-reinforced composites. Fiber bundles mean composites constituted by parallel fibers of identical elastic properties but stochastically distributed breaking thresholds. A fiber fails during the loading process when the local load on it exceeds its breaking threshold. Fiber failures are followed by a redistribution of load on the remaining intact fibers according to the range of interaction in the system. If an infinite range of interaction between the fibers is considered (mean-field approach), then the load of a broken fiber is redistributed among all the intact remaining fibers. The so-called Continuous Damage Model introduced recently [42] is particularly suited to model granular materials since it captures gradual stiffness changes in the system.

In the model of compressed granular packings, force lines formed by particles are represented by an array of lines organized in a square lattice, as illustrated in fig. 5. Let us consider a system with $N$ of such lines with identical stiffness (or Young modulus) $E_0$ but with random failure thresholds $d_i$, $i = 1, 2, \ldots, N$. The failure strength $d_i$ of individual lines is an independent random variable uniformly distributed with probability density $p(d/d_c)$ and an associated cumulative probability $P(d/d_c) = \int_0^d p(x/d_c)dx$, where $d_c$ is the characteristic strength of the force lines [43]. During the compression process, when the local load on a line exceeds its threshold value $d$, the line undergoes a sudden restructuring, becoming stiffer and straighter. The stiffness of the broken line increases after restructuring in a multiplicative manner:

$$E_i \rightarrow a \times E_i, \quad a > 1.$$
This is why the model is also called an “inverted fiber bundle model”: The factor $a$ is no longer in the range $0 < a < 1$, which would imply a reduction of the stiffness of the broken fiber by the factor $a$ [44].

Once a fiber has failed, its damage threshold $d_i$ can either be kept constant for the further failures (quenched disorder), or new failure thresholds of the same distribution can be chosen (annealed disorder). The damage law of the model is illustrated in fig. 7. For modeling the breakings of force chains in granular packings, we choose the second type of disorder shown in fig. 7(b), as it is able to model microscopic rearrangements in the system (redistribution of the grains in the packing). Therefore, after each rearrangement, the force chain gets a new threshold value $d$ from a distribution of the same functional form. On the other side, the characteristic strength $d_c$ of the distribution is increased in a multiplicative way ($d_c \rightarrow d_0 \times d_c$), so that after $k$ rearrangement events, the disorder distribution takes the form

\[ P_k \left( \frac{d}{d_c} \right) = P_0 \left( \frac{d}{d_0^k d_c} \right), \]

where we set $d_c$ to be initially equal to unity. After failure of a line, a certain amount of discharged load has to be taken by the other fibers. For the load redistribution we assume an infinite range of interaction among the force chains (mean-field approach); furthermore, equal strain condition is imposed, which implies that stiffer lines of the system carry more load. At a strain $\epsilon$, the load of the line $i$ that has failed $k(i)$ times reads as

\[ f_i(\epsilon) = E_0 a^{k(i)} \epsilon, \]

where $E_0 a^{k(i)}$ is the actual stiffness of the line $i$. An important parameter of the model is the maximum number of failures, $k_{\text{max}}$, allowed for a single line. In the fiber bundle model, the stiffness of a breaking fiber is decreased after each failure $k = 1, 2, \ldots$, until $k_{\text{max}}$ failures happen and the fiber cannot carry any more load. For the granular packings, when the particles are elastic, $a^{k_{\text{max}}}$ is a finite value related to the stiffness $E_p$ of an individual particle as $E_p = a^{k_{\text{max}}} E_0$. In this case, after all the lines reach the allowed restructuring events number $k_{\text{max}}$, the system will have the same constitutive behaviour as a single contact. The maximum value of possible restructurings, $k_{\text{max}}$, is proportional to the number of contacts, and therefore to $h/l_0$ (see fig. 5). In the following, the initial stiffness $E_0$ of the force lines will be set to unity, so that the stiffness of the line $i$ after $k$ failures reads as $a^{k(i)}$.

The statistics of the microscopic rearrangements in a set of compressed grains due to an external load as defined above can be described by the probability $p_{b_k}(\epsilon)$ that, during the loading of a specimen, an arbitrarily chosen force line at a strain $\epsilon$ breaks precisely
$k$-times. For the \textit{quenched disorder} illustrated in fig. 7(a), we have

\begin{equation}
\rho \mathcal{b}_k(\epsilon) = \begin{cases} 
1 - P_0(\epsilon), & k = 0, \\
P_k(a^{k-1}\epsilon) - P_k(a^k\epsilon), & 1 \leq k \leq k_{\text{max}} - 1, \\
P_{k_{\text{max}}}(a^{k_{\text{max}}-1}\epsilon), & k = k_{\text{max}}.
\end{cases}
\end{equation}

whereas for the \textit{annealed disorder} (fig. 7(b)), which is used to model the compression of the granular packings considered here, $\rho \mathcal{b}_k(\epsilon)$ can be cast in the following form:

\begin{equation}
\rho \mathcal{b}_k(\epsilon) = \begin{cases} 
[1 - P_k(a^k\epsilon)] \prod_{j=0}^{k-1} P_j(a^j\epsilon), & 0 \leq k \leq k_{\text{max}} - 1, \\
\prod_{j=0}^{k_{\text{max}}-1} P_j(a^j\epsilon), & k = k_{\text{max}}.
\end{cases}
\end{equation}

It can be easily seen that the probability equations above fulfill the normalization condition $\sum_{k=0}^{k_{\text{max}}} \rho \mathcal{b}_k(\epsilon) = 1$. With these equations, average quantities of the ensemble of force chains during a loading process can be calculated, as for instance the average load,
or stress $\sigma$ on a line at a given strain $\epsilon$:

\begin{equation}
\sigma = \epsilon \left[ \sum_{k=0}^{k_{\text{max}}} a^k p b_k(\epsilon) \right],
\end{equation}

which provides the macroscopic constitutive behaviour of the model. The expression in the brackets of eq. (11) represents the macroscopic effective stiffness (or Young modulus) of the sample. For example, after one restructuring event, the above defined stress $\sigma$ is written as

\begin{equation}
\sigma = \epsilon [1 - P_0(\epsilon)] + a\epsilon P_0(\epsilon),
\end{equation}

where $P_0(\epsilon)$ and $1 - P_0(\epsilon)$ are the fraction of failed and intact lines, respectively. The total load carried by the intact (failed) lines is thus represented by the first (second) term of eq. (12). Note that in the Fiber Bundle Model, eq. (12) represents the constitutive law for the case in which fibers are allowed to fail only once ($k_{\text{max}} = 1$). Indeed, if we set $a = 0$ in this equation (i.e. broken fibers do not carry any load), the “dry” Fiber Bundle Model [45-48] is recovered, while the case in which $a = 0.5$ corresponds to the so-called “micromechanical model of fiber reinforced ceramic matrix composites” (CMCs), which has been extensively studied in the literature [49-51]. When fibers are allowed to fail more than once, we have to distinguish between quenched and annealed disorder. As mentioned before, here we will consider annealed disorder, since it models the microscopic redistributions of the grains in the packing during compression.

After two restructuring events, eq. (11) can be calculated by using the probability law (10) with $k = 2$, which gives

\begin{equation}
\sigma = \epsilon [1 - P_0(\epsilon)] + a\epsilon P_0(\epsilon)[1 - P_1(a\epsilon)] + a^2\epsilon P_0(\epsilon) P_1(a\epsilon),
\end{equation}

where $P_0(\epsilon)[1 - P_1(a\epsilon)]$ is the fraction of lines that failed only once, and $P_0(\epsilon) P_1(a\epsilon)$ is the fraction of lines that failed already twice. Finally, after $k_{\text{max}}$ restructuring events, the constitutive equation (11) is given by

\begin{equation}
\sigma = \sum_{k=0}^{k_{\text{max}}-1} a^k \epsilon [1 - P_k(a^k\epsilon)] \prod_{j=0}^{k-1} P_j(a^j\epsilon) + a^{k_{\text{max}}} \epsilon \prod_{k=0}^{k_{\text{max}}-1} P_k(a^k\epsilon).
\end{equation}

The probability distributions $P_k$, $P_j$ above follow the relation $P_k(a^k\epsilon) = P_0(\epsilon/(d_0)^k)$ ($d_0 = 1$), which determines the evolution of the system with the restructuring events. Therefore, the ratio $\tau = a/d_0$ is a very important parameter of the model; it decides whether the force chain becomes more fragile ($\tau > 1$) or more ductile ($\tau < 1$) as a result of restructuring. For the case in which $a = d_0$, we have

\begin{equation}
\tau = 1 \implies P_k(a^k\epsilon) = P_0(\epsilon).
\end{equation}
For this particular regime, and in the limit of $k_{\text{max}} \to \infty$ where the particles are hard, eq. (14) can be written as

\begin{equation}
\sigma(\epsilon) = \sum_{k=0}^{k_{\text{max}}-1} a^k \epsilon [1 - P_0(\epsilon)] [P_0(\epsilon)]^k + a^{k_{\text{max}}} \epsilon [P_0(\epsilon)]^{k_{\text{max}}}
= \epsilon [1 - P_0(\epsilon)] \sum_{k=0}^{k_{\text{max}}-1} [aP_0(\epsilon)]^k + \epsilon [aP_0(\epsilon)]^{k_{\text{max}}}.
\end{equation}

It can be seen from eq. (16) that if the maximum number $k_{\text{max}}$ of possible restructuring events goes to infinity, the stress $\sigma$ has finite values only for $aP_0(\epsilon) < 1$. In this case the second term goes to zero, and the constitutive equation takes the form

\begin{equation}
\sigma(\epsilon) = \epsilon [1 - P_0(\epsilon)] \frac{1}{1 - aP_0(\epsilon)}, \quad aP_0(\epsilon) < 1.
\end{equation}

It follows that the stress $\sigma$ diverges when $\epsilon$ approaches a critical value $\epsilon_c$, where $P_0(\epsilon_c) = 1/a$. Expanding $P_0(\epsilon)$ into a Taylor series at $\epsilon_c$ as $P_0(\epsilon) = P_0(\epsilon_c) + p_0(\epsilon_c)(\epsilon_c - \epsilon) + \cdots$, where $p_0(\epsilon) = dP_0/d\epsilon$ is the distribution density corresponding to $P_0(\epsilon)$, and substituting it into eq. (17), the behaviour of $\sigma$ in the vicinity of $\epsilon_c$ reads as

\begin{equation}
\sigma(\epsilon) \approx \frac{1}{ap_0(\epsilon_c)(\epsilon_c - \epsilon)} \sim (\epsilon_c - \epsilon)^{-1}.
\end{equation}

It means that the stress $\sigma$ exhibits a power law divergence when $\epsilon$ approaches the critical value $\epsilon_c$. The critical exponent can be calculated as

\begin{equation}
\nu = \lim_{\epsilon \to \epsilon_c} \frac{\ln \sigma(\epsilon)}{\ln(\epsilon_c - \epsilon)} = -1.
\end{equation}

The value of the exponent is universal; it does not depend on the form of the disorder distribution $P_0$. On the other side, the value of $\epsilon_c$ does depend on the choice of $P_0$. For a uniform distribution $P_0(\epsilon) = \epsilon$, we have $\sigma(\epsilon) = \epsilon [1 - \epsilon](1/a - \epsilon) \to 1 - a\epsilon_c = 0 \Rightarrow \epsilon_c = 1/a$, while for a cumulative Weibull distribution

\begin{equation}
P_k(d) = 1 - \exp \left[ - \left( \frac{d}{d_0} \right)^m \right],
\end{equation}

which has been proved to be a good empirical statistical distribution for modeling solid strength in material science, the critical value $\epsilon_c$ can be calculated, again for $\tau = 1 \Rightarrow a = d_0$, as follows:

\begin{equation}
P_k(a^k \epsilon) = P_0(\epsilon) = 1 - e^{-\epsilon m} \Rightarrow \\
\Rightarrow \sigma(\epsilon) = \epsilon [1 - (1 - e^{-\epsilon m})] \frac{1}{1 - a (1 - e^{-\epsilon m})} \Rightarrow \epsilon_c = \left( \ln \frac{a}{a - 1} \right)^{1/m}.
\end{equation}
Experiments and discrete element simulations [33,39,52] have revealed that the
number of effective force chains increases during the compression process, until it reaches a
saturation value. To incorporate this phenomenological aspect in the model, we write
the number of force chains as \( N(\varepsilon) = N_0 G(\varepsilon) \), where \( N_0 \) is the saturated number of
chains, and the profile \( G(\varepsilon) \) has the property \( G(\varepsilon) \to 1 \) with increasing \( \varepsilon \). The number of
force lines \( dN \) emerging due to an infinitesimal deformation increment from \( \varepsilon \) to \( \varepsilon + d\varepsilon \)
is given by

\[
dN = N_0 g(\varepsilon) d\varepsilon,
\]

where \( g(\varepsilon) = dG(\varepsilon) / d\varepsilon \). A more general constitutive equation of compressed granular
systems can be obtained by taking into account the whole loading history of the system,
as well as possible non-linearities of the contacts between the grains. For this, we write
the load \( f_i(\varepsilon) \) introduced in eq. (8) in a different way: \( f_i(\varepsilon - \varepsilon_0) = E_0 a^{k(i)}(\varepsilon - \varepsilon_0)^\alpha \), where
now \( \alpha \neq 1 \), and the local strain \( \varepsilon - \varepsilon_0 \) is different from the externally imposed strain
value \( \varepsilon \), since it also depends on the initial strain \( \varepsilon_0 \). By introducing the density function
\( g(\varepsilon_0) \) as defined above into eq. (11), the average load or stress \( \sigma \) on a line reads as

\[
\sigma(\varepsilon) = \sum_{k=0}^{k_{\text{max}}} \int_0^\varepsilon (\varepsilon - \varepsilon_0)^\alpha a^k [p_{bk}(\varepsilon - \varepsilon_0)] g(\varepsilon_0) d\varepsilon_0,
\]

where \( E_0 \) is again set to be unity. The probabilities \( p_{bk}(\varepsilon - \varepsilon_0) \) can be calculated with
eq. (10). It follows that the macroscopic constitutive equation of the compressed granular
system, again with the condition \( \tau = 1 \), can be cast into the following expression:

\[
\sigma(\varepsilon) = \sum_{i=0}^{k_{\text{max}}} \int_0^\varepsilon a^i(\varepsilon - \varepsilon_0)^\alpha P_i^k(\varepsilon - \varepsilon_0)[1 - P_0(\varepsilon - \varepsilon_0)] g(\varepsilon_0) d\varepsilon_0 +
\]

\[
+ \int_0^\varepsilon a^{k_{\text{max}}}(\varepsilon - \varepsilon_0)^\alpha P_0^{k_{\text{max}}}(\varepsilon - \varepsilon_0) g(\varepsilon_0) d\varepsilon_0,
\]

which can be numerically calculated by choosing the correct parameters and the appro-
piated form of \( g(\varepsilon_0) \). Figure 8 shows how eq. (24) can be fitted to experimental data
obtained with the apparatus of fig. 5, with the measured force presented as a function
of the relative displacement of the cylinder top. The parameters corresponding to the
plot shown in this figure are: \( E_0 = 4600 \text{ N/m}^2 \), \( d_0 = 3 \text{ N} \) and \( k_{\text{max}} = 70 \). It has been
assumed that the new force chains emerge following an exponential law \( g(\varepsilon) = e^{-\varepsilon/\beta} \),
with \( \beta = 0.01 \). A power law \( F \sim \varepsilon^3 \) is also shown. The emergence of gradual hardening
force chains due to rearrangements of grains inside the container is responsible on the
microscopic level for the strong non-linearity observed macroscopically. If we consider
the number \( N \) of force lines as fixed, analytic results can be obtained also for the statistics
of the restructuring events. Restructuring occurs during the compression process
whenever the local load on a force line exceeds its threshold value. Since loading is per-
formed under strain-controlled conditions, there is no load redistribution among existing
force lines. If the new threshold value assigned to the force line after rearrangement is smaller than the local load, the actual line undergoes successive restructurings until it gets stabilized. The number of steps to reach the stable state defines the size \( s \) of the restructuring event. The number \( n_{k,s} \) of restructuring events of size \( s \) starting in force chains which have already suffered \( k \) restructurings reads as

\[
\frac{n_{k,s}(\epsilon)}{N_0} = \prod_{j=1}^{k-1} P_j(a^j \epsilon) \frac{dP_k(a^k \epsilon)}{d\epsilon} \times \prod_{r=1}^{s-1} P_r(a^r \epsilon) [1 - P_s(a^s \epsilon)],
\]

for \( s \leq k_{\text{max}} - k - 1 \), and

\[
\frac{n_{k,s}(\epsilon)}{N_0} = \prod_{j=1}^{k-1} P_j(a^j \epsilon) \frac{dP_j(a^j \epsilon)}{d\epsilon} \times \prod_{r=1}^{s-1} P_r(a^r \epsilon), \quad \text{for } s = k_{\text{max}} - k.
\]

The conditional probability is normalized by the total number of elements \( N_0 \), and is shown in fig. 9 for several values of \( k \), with \( s = 2 \). For \( \tau = 1 \) we have \( P_k(a^k \epsilon) = P_0(\epsilon) \), and the expression for \( n_{k,s}(\epsilon) \) can be written as follows:

\[
\frac{n_{k,s}(\epsilon)}{N_0} = \begin{cases} 
\frac{dP_0(\epsilon)}{d\epsilon} P_0^{s+k-1}(\epsilon) [1 - P_0(\epsilon)], & s \leq k_{\text{max}} - k - 1, \\
\frac{dP_0(\epsilon)}{d\epsilon}, & s = k_{\text{max}} - k.
\end{cases}
\]
Fig. 9. — Normalized number of restructuring events of size $s = 2$ starting in force chains which have already failed $k$ times. The circles represent the conditional probability obtained experimentally, while the analytic results are shown as full lines.

The number $D(s)$ of events of size $s$ can be determined by integrating over the entire loading history and summing over all possible $k$ values:

$$ (26) \quad D(s) = \sum_{k=0}^{k_{\text{max}}-s-1} \int_0^{\varepsilon_c} \frac{n_{k,s}(\varepsilon)}{N_0} \, d\varepsilon + \int_0^{\varepsilon_c} \frac{n_{k_{\text{max}}-s,s}(\varepsilon)}{N_0} \, d\varepsilon. $$

Substituting expression (25) into the equation above, we get

$$ (27) \quad D(s) = \sum_{k=0}^{k_{\text{max}}-s-1} \int_0^{\varepsilon_c} \frac{dP_0(\varepsilon)}{d\varepsilon} P_0^{s+k-1}(\varepsilon)[1 - P_0(\varepsilon)]d\varepsilon + \int_0^{\varepsilon_c} P_0^{k_{\text{max}}-1}(\varepsilon) \frac{dP_0(\varepsilon)}{d\varepsilon} d\varepsilon $$

$$ = \sum_{k=0}^{k_{\text{max}}-s-1} \int_0^1 P_0^{s+k-1}[1 - P_0]dP_0 + \int_0^1 P_0^{k_{\text{max}}-1}dP_0 \rightarrow $$

$$ \Rightarrow D(s) = s^{-1}, \quad \text{where } 1 \leq s \leq k_{\text{max}}, $$

i.e. the distribution of microscopic restructuring events exhibits a universal power law behaviour with an exponent 1, which is completely independent of the disorder distribution. Numerical simulations revealed that this behaviour holds even if the gradual creation of force chains is taken into account, i.e. when the system is described by eq. (24). In fig. 10 we show the statistics of restructuring events obtained by Monte Carlo simulations performed with the assumption $N(\varepsilon) = N_0(1 - e^{-\varepsilon/\beta})$. A sequence of local events of different sizes, that occurred during the loading process, is shown in the inset of this
Fig. 10. – The inset shows local events of size $s$ obtained from a Monte Carlo simulation of strain controlled loading. The distribution $D(s)$ is presented in a double logarithmic plot in the main figure.

While the corresponding distribution $D(s)$ is presented in the main plot. As we can see from the comparison of this figure with fig. 6, the theoretical results on the statistics of the restructuring events are in a very good quantitative agreement with the experimental findings. It is important to emphasize that the functional form of $D(s)$, as well as the value of the exponent of the power law (27), have their origin mainly in the local behaviour of the restructuring processes, as there is no load redistribution among intact lines. The excellent agreement of the results presented here with experimental data indicates that this is probably the microscopic mechanism responsible for the power law statistics of acoustic signals observed experimentally.

4. Conclusions

In this lecture, we have seen several aspects of the non-trivial behaviour of granular materials. A short review of the research on grains was presented, and we concentrated on the phenomenology of granular media in the static form. The last section of these lecture notes was dedicated to the model of percolating force chains in compressed granular packings, recently published [41]. Other sections could be added for the study of grains in the fluidized form, but this is rather the subject for another lecture. It has been shown that many open problems on granular media remain to be investigated in the future. We hope very much that this pedagogical enterprise will help to motivate young researchers to take up some of these challenges.
We acknowledge S. McNAMARA for a critical reading in this manuscript and helpful comments. E. J. R. Parteli acknowledges a fellowship from CAPES, Brasília/Brazil.

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