Simulation of the sedimentation of a falling oblate ellipsoid

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Abstract

We present a numerical study of the dynamics of one falling oblate ellipsoid particle in a viscous fluid, in three dimensions, using a constrained-force technique (Doctoral Thesis, Stuttgart University, 2000, Phys. Rev. E 61 (2000) 7146, J. Eng. Math. 41 (2001) 221). We study the dynamical behavior for a typical downward motion. Three types of falling motions are established: steady-falling, periodic oscillations and chaotic oscillations. For the periodic and steady-falling regime we find a similarity law derived from the invariance of the Reynolds and Froude number. In the chaotic regime the trajectory of the oblate ellipsoid is characterized by a high sensitivity to tiny variations in the initial orientation. The Lyapunov exponent is \( \lambda = 0.052 \pm 0.005 \). A phase diagram is presented and compared to the results of Field et al. (Nature (London) 388 (1997) 252). The transition from oscillatory to steady-falling occurs at \( Re_c = 355 \), where the transient time of oscillation in the steady-falling regime tends to infinity, beyond this value the system is oscillatory. The transient time has a power law divergence at \( Re_c \) with an exponent of 0.5. The transition from steady-falling to chaotic regime becomes abrupt, for an aspect-ratio \( \Delta r_c \approx 0.22 \).

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1. Introduction

Newton showed that bodies fall on earth driven by a constant acceleration. But despite gravity’s undeniable attraction, not all falling objects travel downwards on straight trajectories. The tree leaves flutter to the ground in the autumn, exhibiting a

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complex motion and refusing to follow the shortest path. The consideration of the fluid, becomes a very complicated and nonlinear interaction. In spite of the pioneering effort made by Maxwell [1], who was the first in consider the fluid-object interaction, the general problem remains without solution.

In the last decade theoretical work has been done elucidating with success some aspects of falling bodies. Aref and Jones [2] find chaotic behavior in the Kirchhoff equations. Tanabe and Kaneko [3], using a phenomenological model describe five falling regimes. Mahadevan [4] using a more elaborated model, finds tumbling and drifting motion. Field et al. [5], obtain experimental evidence for chaotic behavior. Again, Mahadevan et al. [6], in an experiment find scaling behavior between the tumbling frequency and the thickness $d$ and width $w$ of dropping horizontal cards. Belmonte et al. [7], also in an experimental work, show the dependence of the types of motion on the Froude number.

Given the difficulties to study this problem theoretically and experimentally, we took a computational approach simulating the falling of one oblate ellipsoid in a viscous fluid in a three dimensional container. We organize the paper in the following manner. In Section 2 we give a review over the model that we use. In Section 3.1 we present the dependence on kinematic viscosity and the characteristic time in the trajectories. Section 3.2 presents the similarity law, for the steady-falling regime. In Section 3.3 we show the similarity for the oscillatory regime. In Section 3.4 we show the chaotic regime. In Section 3.5 the parameter phase space is depicted and compared to the results of reference [5]. The Section 3.6 presents the transition from steady-falling to oscillatory regime and the transition from steady-falling to chaotic regime. Section 4 summarizes and discusses possible further applications.

2. Model

The general idea, proposed by Fogelson and Peskin [8], is to work with a simple grid for the resolution of the fluid motion at all times and represent the particles not as boundary conditions to the fluid, but by a volume force or Lagrange multipliers in the Navier–Stokes equations.

This technique was developed in the work of Schwarz et al. [9,10], Kuusela et al. [11].

The motion of the fluid is described by the dimensionless and incompressible Navier–Stokes equations:

$$ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{u} + \vec{f}, $$

$$ \nabla \cdot \vec{u} = 0. $$

(1)

(2)

Here $p$ and $\vec{u}$ are the pressure and the velocity of the fluid, respectively, and $\vec{f}$ is an external force. The Reynolds number $Re = U(2R_{Mf}/(v))$, where $U$ is the mean vertical oblate ellipsoid velocity, $2R_M$ is the largest oblate ellipsoid’s diameter, $\rho_f$ the density.
and \( v = \mu/\rho_f \) the kinematic viscosity (\( \mu \) the shear viscosity). The Froude number is 
\[
Fr = \frac{U^2}{(g(2R_M))}
\]
with \( g \) the gravity.

Eq. (1) is discretized on a regular, marker-and-cell mesh to second order precision in space. For the time stepping, we employ an operator-splitting-technique which is explicit and accurate to first order. The suspending fluid is subjected to no-slip boundary conditions at the surface of the suspended particles. More details of the solution procedure are presented in Refs. [9–11]

An oblate ellipsoid is represented by a rigid template connected to fluid tracer particles, which are moving on the trajectories of the adjacent fluid. The connection is made by using the body force term, in the Navier–Stokes equations, as constraints on the fluid such to describe the oblate ellipsoid.

The force density \( \mathbf{f}^c \), is chosen elastic with a spring constant that guarantees that the elongation, \( |\mathbf{a}| \) Eqs. (3), (4), remains small against the grid spacing at all times [9], and it is zero in the exterior of the region outside the oblate ellipsoid. We can define this force density \( \mathbf{f}^c \) as
\[
\mathbf{f}^c = f^c(x_{ij} \hat{a}(x_{ij})) = -k\mathbf{a}(x_{ij}) ,
\]
where \( x_{ij} \) is the displacement field of the separation between the markers \( i \) and their corresponding reference point \( j \). The stiffness constant \( k \), must be chosen large enough so that \( |\mathbf{a}(x_{ij})| \ll h \), \( h \) size grid, holds for all iterations.

In general the displacement field \( \mathbf{a}(x_{ij}) \) is defined as
\[
\mathbf{a}_i(x_{ij}) = \overrightarrow{x}_{ij} = \overrightarrow{x}_{ij} - \overrightarrow{x}_{ij} .
\]

The vector \( \overrightarrow{x}_{ij} \) is the position of a fluid tracer, whose motion is determined by the fluid local velocity, i.e.,
\[
\overrightarrow{x}_{ij} = \overrightarrow{u}(x_{ij}) .
\]
The \( \overrightarrow{x}_{ij} \) are the reference points associated to a template having the shape of the physical particle:
\[
\overrightarrow{x}_{ij} = \overrightarrow{x}_{ij} - O_i(t) \cdot \overrightarrow{r}_{ij} .
\]
Here \( \overrightarrow{x}_{ij} \) is the center of mass of the template, \( O_i(t) \) is the rotation matrix that describes the present orientation of the oblate ellipsoid and \( \overrightarrow{r}_{ij} \) denote the initial position of the reference points with respect to the center of mass. For the quaternion formulation of the rotation, we use the technique described in Ref. [12].

The equations of motion of the particle template are
\[
\dot{\mathbf{U}} = \frac{\mathbf{F}}{M}
\]
and
\[
\mathbf{I} \dot{\mathbf{\Omega}} = \mathbf{T} ,
\]
where \( M \) is the mass of the template particle; \( \mathbf{U} \) and \( \mathbf{\Omega} \) are the linear and angular velocities of the template particle, respectively; \( \mathbf{I} \) is the moment of inertia tensor for the oblate ellipsoid with uniform density and has by symmetry only three non-zero
elements, $I_{11}I_{22}I_{33}$ where $I_{11} > I_{22} = I_{33}$, where $I_{33} = I_{oblate}$ [14]; and $\vec{T}$ is the torque, [9,11].

A velocity-Verlet integrator [13] serves to integrate the equations of motion for the translation and a Gear-predictor integrator [12] for the rotation on the template:

$$\vec{F} = -Mg\vec{j} + \rho_f Vg\vec{j} + \sum_i \vec{f}_i^c,$$

where $\vec{j}$ is the unit vector along the vertical.

$$\vec{T} = \sum_i (\vec{x}_i - \vec{x}_{cm}) \times \vec{f}_i^c$$

with respect to the template's center of mass $\vec{x}_{cm}$.

The boundary conditions at the container wall are zero for the normal velocity component of the fluid and no-slip condition for the tangential component, because the walls are assumed impenetrable, [9]. The interaction between the oblate ellipsoid and the walls is defined through a contact force, [11], where the walls are treated as a particle with infinite mass and infinite radius.

For all our simulations, and in order to reduce the parameter space we fix the ratio between the density of the oblate ellipsoid and the fluid as $\rho_{oblate}/\rho_{fluid} = 3.5$ in the system. The system units are given so that the smaller oblate ellipsoid radii, the Stokes velocity, and the density of the fluid are equal to unity. The container is a squared base $L_{hor} \times L_{hor}$, with height $L_{ver}$, and the ellipsoid falling height is $h_0$. We take the angle between the ellipsoid's normal and the vertical, to define the initial orientation $\theta_0$, in consequence a horizontal ellipsoid will have $\theta_0 = 0$.

The geometry of the oblate ellipsoid is characterized by $\Delta r$, its aspect-ratio, defined as the ratio of the minor radius $R_m$ to the major radius $R_M$:

$$\Delta r = \frac{R_m}{R_M}.$$  

We define $T_s$, the Stokes time for an equivalent sedimenting sphere as

$$T_s = \frac{R_{equ}}{v_s},$$

where $R_{equ}$ is the radius of a sphere with the same volume of an oblate ellipsoid, and is defined as

$$R_{equ} = (R_M^2 R_m)^{1/3}$$

and $v_s$ is the Stokes velocity

$$v_s = \frac{2gR_{equ}^2 (\rho_{ell} - \rho_f)}{9

\nu}.$$  

In order to adimensionalize the kinematic viscosity we use

$$v_s = \frac{T_s}{R_{equ}^2}.$$
3. Results

We found three different motions in our simulations: steady-falling, side-to-side periodic-oscillation known as ‘flutter’ Ref. [7], and a chaotic motion as shown in Fig. 1. The above phenomenology can be compared to the work of Ref. [5], for the case of dropping disks. In general, the trajectories depend strongly on the initial conditions and the properties of the system (oblate ellipsoid’s orientation $\theta_0$, kinematic viscosity $v$ and the oblate ellipsoid’s aspect-ratio $\Delta r$, etc).

Fig. 1. The top figure shows the bidimensional vertical trajectory for: (a1) steady-falling, with initial conditions $\theta_0=26.6^\circ$, $\Delta r=0.25$, $v=0.033$ and $h_0=240$; (b1) periodic-oscillation, with initial conditions: $v=0.025$, $d_0=240$, $\Delta r=0.133$ and $\theta_0=63.4^\circ$. (c1) chaotic motion, with initial conditions $h_0=96$, $\Delta r=0.25$, $v=0.033$ and (a)$\theta_0=26.6^\circ$. The bottom figure presents the same trajectories a2, b2, c2 but for the perpendicular vertical plane.
3.1. Steady-falling oblate ellipsoid: dependence on the kinematic viscosity

In Fig. 2 we show the decreasing exponential amplitude of oscillation in time, for different values of the kinematic viscosity, so that it gives a straight line in a log-linear plot. In all cases the logarithm of the amplitude behaves as a decays linearly in time, and plays the same role as in a damped harmonic oscillator. The coefficient that dimensionalize the argument inside the exponential, is called the characteristic time $T/T_s$, and decays linearly with the kinematic viscosity $v/v_s$, insert Fig. 2.

All the points fall onto relation $T/T^* = -0.03v/v_0 + 5.5$, this is plotted in the inset of Fig. 2. The results shown in Fig. 2, clearly exhibit the exponentially decreasing amplitude of the oscillation and the role of the viscosity as a damping factor and which determines the decay rate of the vertical position and velocity.

3.2. Steady-falling oblate ellipsoid: similarity law

For large velocities it is well known that the inertial drag [7] is given by

$$F_d = C_{\rho f} V^2 S,$$

where $C$ is the form factor of the inertial drag, $[7]$, $S$ is the cross-sectional area of the oblate ellipsoid, $\rho_f$ the fluid density, and $V$ the ellipsoid velocity. The weight of the
Table 1  
Transformations rules

<table>
<thead>
<tr>
<th>System $L$</th>
<th>$\rightarrow$</th>
<th>System $L'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{\text{hor}}$</td>
<td>$\rightarrow$</td>
<td>$n \ast L_{\text{hor}}$</td>
</tr>
<tr>
<td>$R_M$</td>
<td>$\rightarrow$</td>
<td>$n \ast R_M$</td>
</tr>
<tr>
<td>$R_m$</td>
<td>$\rightarrow$</td>
<td>$n \ast R_m$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\rightarrow$</td>
<td>$\frac{g}{4n}$</td>
</tr>
</tbody>
</table>

Table 2  
Period and velocity applying the transformation rules in Table 1

<table>
<thead>
<tr>
<th>System $L$</th>
<th>$\rightarrow$</th>
<th>System $L'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^*$</td>
<td>$\rightarrow$</td>
<td>$n \ast T^*$</td>
</tr>
<tr>
<td>$V_{\text{hor}}$</td>
<td>$\rightarrow$</td>
<td>$\frac{V_{\text{hor}}}{n}$</td>
</tr>
<tr>
<td>$V_{\text{vert}}$</td>
<td>$\rightarrow$</td>
<td>$\frac{V_{\text{vert}}}{n}$</td>
</tr>
</tbody>
</table>

Oblate ellipsoid is proportional to

$$F_i \approx \rho_{\text{ell}} R_M^2 R_m g,$$  \hspace{1cm} (17)

where $\rho_{\text{ell}}$ the ellipsoid density and $R_M$, $R_m$ the minor and major radius, respectively. The terminal downward velocity is determined by the equilibrium between these two forces, and since we are fixing both densities, the terminal downward velocity is given by

$$F_i = F_d \quad \Rightarrow \quad V \sim \sqrt{g R_m}.$$  \hspace{1cm} (18)

We suppose that in this regime the characteristic period of oscillation in the system is given by the lateral dimensions of the container $L_{\text{hor}} = 25$, and by means of our results, there is no effect of the falling height (vertical dimension). Therefore we assume that the oscillation period in the system must change as

$$T^* \sim \frac{L_{\text{hor}}}{V}.$$  \hspace{1cm} (19)

The dynamics of the system, in general, depends on the Reynolds and the Froude numbers, [7]. We can rescale the parameters of a system of size $L$ to a system of size $L'$ through:

The transformations presented in Table 1 keep the Froude and Reynolds numbers constant and the dynamics in the two systems become equivalent if the velocity components change as in Table 2. As a consequence of this transformation, the dynamics that should be observed in the system $L'$, $n = 2$ (Table 2), is the change in the oblate ellipsoid period, and in the vertical and horizontal velocity as which is shown in Fig. 3.

In Fig. 3 we present the superposition of the vertical velocity, which is obtained applying to $L'$, $n = 2, 4$ the inverse transformation given in Table 1. The three curve coincide quite well, supporting the scaling.
Fig. 3. Initial conditions in the system: $\theta_0 = 26.6^\circ$, $h_0 = 220$, $\Delta r = 0.15$ and kinematic viscosity $\nu = 0.083$. In the figure we plot the vertical velocity against time in both systems $L$, (solid line), $L'$ ($n=2$, dash-dotted line) and $L'$ ($n = 4$, dashed line). The superposition is carried out applying the inverse transformation described in Table 1.

3.3. Periodic behavior for the oblate ellipsoid: similarity law

Fig. 4 shows the vertical velocity against time for the three systems $(L)$, $(L', n = 2)$ and $(L', n = 4)$, respectively. We apply the transformation rules of Table 1, and as in the case of the steady falling regime, the dynamics for the systems $L$, $L'$, in the oscillatory regime, are related by a similarity law.

3.4. Chaotic regime: sensitivity to a change in the initial orientation

We can ask what is the sensitivity of the oblate ellipsoid to tiny changes in the initial orientation. In order to get more sensitivity we have incremented the falling height to $h_0 = 415$. The resulting trajectories for four slightly different initial orientations in the vertical plane are presented in Fig. 5. In this regime the system presents a high sensitivity to the initial orientation condition. For the four trajectories the relative initial angular variation is $\Delta \theta_0 = 10^{-3}$. This tiny variation produces a significant change in the shape of all the curves. A small change in the initial orientation results in large changes in position and velocity.

We investigate quantitatively this sensitivity by studying the increment in the Euclidean distance $d_{p1p2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, between the curves (a) and (c)
Fig. 4. Initial conditions in the system. $\theta_0 = 26.6^\circ$, $h_0 = 86$, $\Delta r = 0.15$ and Kinematic viscosity $v = 0.029$. In the figure we plot the vertical velocity against time in both systems $L$ (solid line), $L'$ (dash-dotted line) and $L'$, $n = 4$ (dashed line) and the superposition performed applying the inverse transformation of Table 1.

Fig. 5. Initial conditions $h_0 = 166$, $\Delta r = 0.25$, $v = 0.033$ and tiny variations of the initial orientation (a) $\theta_0 = 45.384^\circ$, (b) $\theta_0 = 45.033^\circ$, (c) $\theta_0 = 44.981^\circ$, (d) $\theta_0 = 44.976^\circ$.

presented in Fig. 5. Fig. 6IV, shows that the distance between nearby points has an overall exponential time dependence $d(t) \sim \exp(\lambda t)$ and the fit gives an estimate for the Lyapunov exponent $\lambda = 0.052 \pm 0.005$. The positivity of the Lyapunov exponent is a clear indication for chaos.
Due to this dependence on small changes in the initial orientation, we proceed to use as a tool of diagnosis, the Fourier power spectrum of the time series of the horizontal coordinate $x(t), x(t + δt), x(t + 2 * δt)\ldots$, with $δt = 0.053566$. A broad spectrum of frequencies appears, as shown in Fig. 7I, indicating chaotic motion.

The autocorrelation function, for the same time series (see Fig. 7I), decreases linearly with time. The points are not independent of each other and self similarity is present in the data.

In Fig 7II, we present a Poincaré sections $(p_x, x)$, corresponding to trajectory (a) in Fig. 5(a), and which are quite irregular.

The orbits are quasi-periodic in the sense that they pass repeatedly and irregularly through the whole domain without ever closing on themselves, and without any particular time period associated with successive passages.

3.5. Phase diagram

We explore the phase space in the dimensionless moment of inertia $I^*$, which is the ratio of the moment of inertia of an oblate ellipsoid around its principal axis to the moment of inertia of a sphere of liquid with the same diameter and the Reynolds number $Re$. We do a similar analysis for our results as in the work of Field et al. [5]. It is important to remark that the mentioned experiment was for a falling disk, with small aspect-ratio, and we expect that the dynamics of the system will be close to that of an oblate ellipsoid.
The definitions of the dimensionless variables for our system are

\[
I^* = \frac{I_{oblate}}{I_{sphere}} = \frac{5}{4} \frac{r_m}{r_M} \frac{\rho_{oblate}}{\rho_{fluid}} = \frac{5}{4} \frac{\rho_{oblate}}{\rho_{fluid}} \Delta r .
\]  

Fig. 8 right, shows our results in a log–log scale. At low values of \(I^*\) (0.3–0.9) and small Reynolds number (high kinematic viscosity), the left-down corner of the diagram, the motion is overdamped and the oblate ellipsoid drops to the bottom container without any oscillation, we begin the steady-falling regime. If the Reynolds number increases (\(Re \geq 100\)), fixing the moment of inertia, the trajectory is composed of successive oscillations that will decrease in amplitude until the oblate ellipsoid finally comes to stop at the bottom part of the container, this is called steady-falling regime.

For small values of \(I^* \ll 1\), we have a flattened ellipsoid, and Reynolds number (\(Re \geq 400\)), the trajectory, velocity and orientation are characterized by oscillations that repeat at equal intervals of time and space, we are in the oscillatory regime.
As we increase $I^*$, the object will become a sphere slightly flattened at the poles, and its dynamics becomes sensitive to smaller variations in the initial orientation, exhibiting a chaotic trajectory.

If we compare our diagram with the experimental results obtained by Field et al. Ref. [5] (Fig. 8 left), we can see that in both pictures, the same distribution of the regimes. We can say the two diagrams are similar, but, with the difference, that the tumbling regime in Field's diagram is not present in our results.

The coexistence of the dynamical phases, explained above, is independent of the ellipsoid initial orientation in our simulations.

3.6. Transition from steady-falling to oscillatory regime

In Fig. 9 we show the behavior of the characteristic time $T^*/T$, adimensionalized using Eq. (15), as we increase the Reynolds number $Re$, going to zero at $Re_c \approx 355$, in Fig. 8. Beyond this point we find the oscillatory regime, that behaves like a steady-falling regime with an infinite characteristic time. Therefore, we can consider $T^*/T$ as the order parameter and the control parameter is the Reynolds number for this transition. This transition is like a second order phase transition. The inset exhibits the power-law behavior with a critical exponent $\approx 0.5$. In the case of the upper part of the transition, in Fig. 8, the variation in the oblate ellipsoid aspect-ratio implies the change from steady-falling to oscillatory regime.
Fig. 9. The order parameter $T^*/T$ vs. the control parameter $Re$, with $Re_c \approx 355$. The inset shows the power-law behavior with an exponent close to 0.5.

Fig. 10. The order parameter $T/T^*$ vs. the control parameter $\Delta r$, with $\Delta r_c \approx 0.22$.

3.7. Transition from steady-falling to chaotic regime

The transition from steady-falling to chaotic regime is presented in Fig. 10. We use the order parameter $T/T^*$, i.e., the inverse of the one used before in order to
describe the transition, and the control parameter is the aspect-ratio $\Delta r$. At $\Delta r_c$ a finite jump in the order parameter is observed. The characteristic time $T/T^*$ disappear due to the non-regular oscillations that are very sensitive to small variations in the initial orientation. This transition seems to be therefore of first order.

4. Conclusions and outlook

The motion of a single oblate ellipsoid settling in a fluid in a three dimensional container has been studied. We found three basic regimes for the dynamics of the system (steady-falling, oscillatory, and chaotic). The steady-falling and the periodic motion exhibit a similar physical behavior as observed for flattened bodies [5,7]. With the exception that the tumbling motion is missing in our simulations.

For the steady-falling and oscillatory regime we obtain a similarity law expressed in Tables 1, 2, which is a direct consequence of the invariance of the Reynolds Froude numbers. An experimental work validating this similarity law is needed.

The periodic behavior in our simulations is found for ($Re \sim 500$), and small ($J^* \leq 0.5$). The separation between the spatial trajectories of the falling oblate ellipsoid diverges for small variations in the initial orientation $\Theta_0$, and grows exponentially in time. The value found for the Lyapunov exponent is $\lambda = 0.052 \pm 0.005$.

The chaotic behavior is present for $J^* \geq 1$ and in the entire range of Reynolds numbers used in the simulation.

The construction of the phase diagram shows three well-defined dynamical regions as in the case of Ref. [5]. But with the difference that the chaotic behavior in the above reference is associated with a transition to chaos through intermittency for which we have no indication in our simulations. The phase diagram is independent on the initial orientation.

Our simulations show that the transition for steady-falling to oscillatory, and the transition from steady-falling to chaotic regime can be understood as second and first order phase transition, respectively, the characteristic transient time being the order parameter.

More work to better understand the role of the fluid pressure and velocity fields should be realized in the future.

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