Self-organized Criticality on Small World Networks

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Abstract

We study the BTW-height model of self-organized criticality on a square lattice with some long range connections giving to the lattice the character of small world network. We find that as function of the fraction p of long ranged onds the power law of the avalanche size and lifetime distri ution changes following a crossover scaling law with crossover exponents 2/3 and 1 for size and lifetime respectively.

Small world networks have recently been observed to exist in many physical, biological and social cases [1, 2, 3, 4, 5]. A simple minded example is the structure of the neo-cortex and the network of acquaintances in certain societies. Another ubiquitous phenomenon (SOC) [6, 7, 8] in nature and society is self-organized criticality, i.e. the appearance of avalanches of all sizes without characteristic scale over a certain range. Easily one can imagine cases in which both of these phenomena occur simultaneously, that means that we find an avalanche type spreading on a sparsely long-range connected network. As an example we propose the spreading of neural information inside the cervical cortex which is due to the threshold behaviour given by the firing rule which automatically induces avalanches of synapses. Another example one can imagine are societies where each individual has a certain threshold of endurance. Let us take the example of Peter Grassberger for the classical SOC-model [6] of office clerks moving sheets of papers from desk to desk, as illustrated in Peter Bak's book [7] in Fig. 13 and admit that the clerks do not sit on a square lattice but have a more realistic connectivity in their work relation (typically small world network behaviour). Many other examples of this kind can be thought of.

In this short paper we want to present a model for self-organized criticality on a graph having the properties of small world networks. In fact, we consider the classical height model of Bak, Tang and Wiesenfeld (BTW) [6] on a square lattice which has been "rewired" for a certain fraction of bonds p which are chosen of arbitrary range. For that purpose we take a square lattice of linear size L and all bonds present between nearest neighbours sites. Then we choose randomly two sites of the system and place a bond between them (which can therefore be of any length smaller than $\sqrt{2}L$). In order to keep the coordination of the sites on average to be four, one of the smaller bonds going to a neighbouring site of one of the end points of one long bond is removed. This procedure repeats until a fraction p of all bonds has been replaced by long range connections. This type of graph is not the same as the one used by most authors working on small world networks, because the underlying short range lattice is not a linear chain but a square lattice. Nevertheless the properties should qualitatively be the same: large world behaviour for large p and a small world behaviour for small p. The situation $p = 0$ corresponds to the simple square lattice and $p = 1$ to random graph with average coordination 4 and long range connections (Viana-Bray) on which one would expect mean-field behaviour.

On each site of the lattice we place an integer value h_i less than a threshold $h_c = 6$. At each time step randomly one site i is chosen and its value h_i is increased by unity, i.e. a unit mass is added to it.

When the value of the height h_i having q neighbours reaches the threshold h_c, it topples, i.e. it distributes mass equally to its neighbours:

$$\begin{aligned} h_i &\rightarrow h_i - q \\ h_j &\rightarrow h_j + 1 \end{aligned} \quad (1)$$

where j goes over all q neighbors of site i. We see that eq. (1) preserves the mass (the sum of all heights).

We have studied the statistics of avalanches monitoring as well their size s (number of sites that toppled at least once) as their lifetime t, that means the number of time step an avalanche lives. The quantities $n(s)$ and $n(t)$ denote the avalanche size and life time distributions. For the case $p = 0$, that means the classical BTW-model on the square lattice it is known that asymptotically

$$\begin{aligned} n(s) &\sim s^{-\sigma}, \\ n(t) &\sim t^{-\tau} \end{aligned} \quad (2)$$

with $\sigma = 1.0$ and $\tau = 0.5$ in two dimensions.

It is our aim here to investigate what happens for different values of p. To this purpose we have analysed square lattices of size $L = 50, 100$ and 200 and a range of p values from $p = 0$ to $p = 0.5$. The lattice has spiral boundary conditions in one direction and two open boundaries at top and bottom, where mass in excess can flow out of the system. In each configuration we have injected randomly 2500 particles of unit mass, letting each time the avalanche proceed until no site had a value of $h > h_c$. The data were averaged over 200, 50 and 10 configurations for $L = 50, 100$ and 200 respectively.

In Fig. 1 we see the distribution of avalanche sizes for different system sizes and two different values of $p, 0.1$ and 0.3 in a double logarithmic plot. We see that the data follow a straight line for nearly two decades indicating that we still find SOC-behaviour. The slope of the straight line gives us the exponent σ. In Fig. 2 we see the corresponding figures for the distribution of life times. Again the data show power law behaviour and the slope gives the exponent τ. We observe that the exponents σ and τ depend on p but they do not appear to depend on L.

In Fig. 3 we see the dependence of σ and τ on p as obtained from our simulations. We see that for $p \rightarrow 0$ we obtain the classical result on the square lattice of BTW $\sigma = 1$, $\tau = 1/2$ [6]. For p close to unity the values of the exponents convert to the meanfield values $\sigma = 3/2$ and $\tau = 1$ [9, 10, 11]. This is of course not surprising and the question is if the continuous change of the exponent σ and τ is an intrinsic continuous line of critical points or, if we have here a crossover phenomenon as it appears for instance in magnetic models that interpolate between 2 and 3 dimensions. For that purpose we tried for both the size and the lifetime avalanche data in Fig. 4, a data collapse of all the distributions for different values of p following the classical crossover scaling

$$n_p(s) = s^{-\sigma}\mathcal{F}(sp^{\phi}) \tag{3}$$

$$n_p(t) = t^{-\tau}\mathcal{G}(tp^{\psi})$$

where $\mathcal{F}$ and $\mathcal{G}$ are scaling functions and ϕ and ψ are universal crossover exponents. We see from Fig. 4 that a collapse of the data works reasonably well yielding crossover exponents of $\phi = 0.65 \pm 0.1$ for the avalanche size distribution and $\psi = 1.0 \pm 0.1$ for the lifetime distribution.

By studying the BTW-model on a small world square network we observed a crossover to meanfield behaviour following a crossover scaling law of eq. (3). It would be interesting to see if the same occurs for the simpler Manna-model and we have heard calculations are already under way [12].

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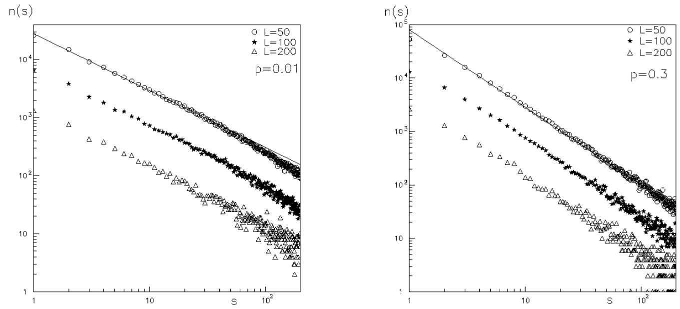

Figure 1: Log-log plot of the distribution of avalanche size $n(s)$ as function of s for different system size L and for $p = 0.01(a)$ and $p = 0.3(b)$. The continuous lines indicate the slopes 0.98 and 1.45 for (a) and (b) respectively. The statistics is of 2500 particles injected in 200,50 and 10 configurations for $L = 50, 100$ and 200 respectively.

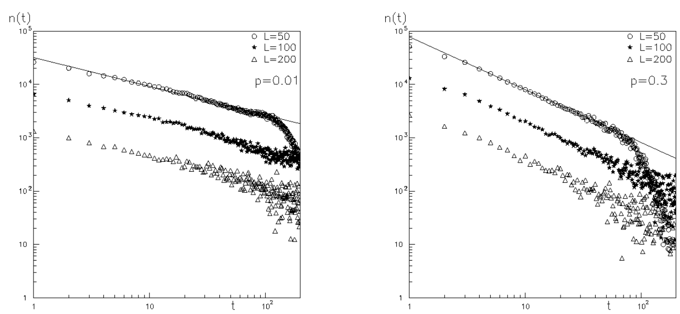

Figure 2: Log-log plot of the distribution of avalanche lifetime $n(t)$ as function of t for different system size L and for $p = 0.01(a)$ and $p = 0.3(b)$. The continuous lines indicate the slopes 0.54 and 0.99 for (a) and (b) respectively. The statistics is the same as in Fig. 1.

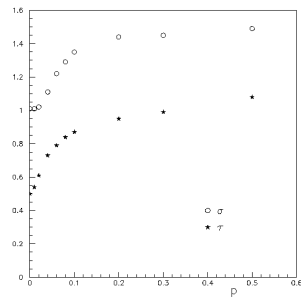

Figure 3: Critical exponents σ and τ for the distribution of avalanche size and lifetime as function of p. The exponent are calculated for the system size $L = 50$.

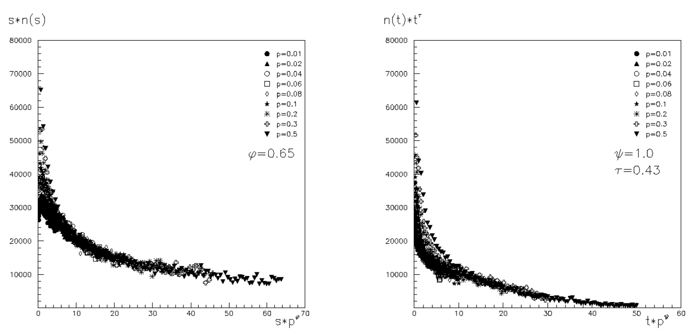

Figure 4: Data collapse of the distributions of avalanche size $n(s)$ and lifetime $n(t)$ for different values of p and $L = 50$. (a) The plot of the quantity $n(s)s^{\sigma}$ for $\sigma = 1.0$ as function of sp^{ϕ} gives a value for the crossover exponent $\phi = 0.65 \pm 0.1$. (b) The analogous plot of the quantity $n(t)t^{\tau}$ for $\tau = 0.5$ as function of tp^{ψ} gives a value for the crossover exponent $\psi = 1.0 \pm 0.1$.