

SIMULATIONS OF GRANULAR MEDIA

H.J. HERRMANN AND M. MÜLLER

*Institute for Computer Applications 1, University of Stuttgart, Pfaffenwaldring 27, 70569
Stuttgart, Germany*

E-mail: {hans, matthias}@ica1.uni-stuttgart.de

For the last ten years there has been an enormous progress in the simulation of granular media like sand or powders. These simulations consist in simulating trajectories of each particle individually. Essentially one has to solve the Newton's equations including the effects of Coulomb friction and the physics occurring at a collision. But the details of the trajectories are not important for the collective behaviour. Therefore simplifications are introduced on the smallest scales.

I will introduce various methods like molecular dynamics that are used to simulate large amounts of particles (over 10^9). Some of these methods are based on the exploitation of parallelisation and metacomputing. Other approaches are more stochastic (DSMC Direct Simulation Monte Carlo) which simplify the calculation of collisions, positions and collision times. Very successful has been also the use of cellular automata which have been able to predict details such as the logarithmic tale of sand heaps. I will also discuss numerical techniques used for the surrounding fluid. This can be water in the case of sedimentation or air when one studies the formation of dunes in the desert. The calculation of velocity and pressure field of the fluid are done using multigrid techniques on parallel computers.

We will compare the performance of the various techniques and show some benchmarks on the dependence on the size of the system, the density of particles and the number of processors used.

1 Introduction

Many rather astonishing phenomena are known to occur when granular materials like sand or powders move ^{1,2,3,4}. Examples are the so-called "Brazil nut" segregation ^{5,6,7,8}, heap formation under vibration ^{9,10,11}, density waves emitted from outlets ¹², and $1/f$ noise in the power spectra of local forces ¹³. All these effects originate in the ability of granular materials to form a hybrid state between a fluid and a solid: When the density exceeds a certain value, the critical dilatancy ^{14,15}, it is resistant to shear, like solids, while below this density it will "fluidify". This fluidified state can be rather complex, specially in the presence of density fluctuations and density gradients.

In order to formalize and quantify the complicated rheology of granular media various attempts have been made. Continuum equations of motion, a kinetic theory ^{3,16}, and thermodynamic formulations ^{17,18,19} have been proposed. But many of the above mentioned effects, such as size segregation or density fluctuations, have so far eluded a satisfactory explanation. This is because it is very difficult to incorporate into these theories static friction, local rotations and other relevant microscopic mechanisms.

To gain a better understanding of the rheological effects of granular media it is therefore very useful to do computer simulations ^{20,21}. For over a decade discrete methods have been used where instead of a continuum one treats the granular material as an assemblage of particles interacting through their contacts. This technique was introduced by Cundall ²² to study the motion of rock masses. Since

then it has been applied to statistical micromechanics^{23,24}, constitutive behaviour of granular soils²⁵, creep of soils²⁶, analysis of rock-support interaction²⁷ and other applications of soil mechanics²⁸. These techniques have also been applied to model size segregation^{8,29}, outflow from a hopper^{30,31}, shear flow³² and flow down an inclined chute³³.

In the following we will discuss these techniques. In particular we will present Molecular Dynamics (MD)^{34,35} simulations of inelastic particles with an additional shear friction.

2 Molecular Dynamics Technique

As opposed to usual molecular gases the elementary units of granular materials are mesoscopic grains each consisting of many atoms ($10^{15} - 10^{25}$). When these objects interact (collide) the attractive potentials of the individual atoms are unimportant and completely different mechanisms must be considered. It is important that on a microscopic scale the surface of the grains is rough. Solid friction is the immediate consequence: When two touching grains are at rest with respect to each other a finite force F_s is needed to trigger relative motion (*static friction*), while moving against each other a finite force F_d is needed to maintain the motion (*dynamic friction*). $F_d < F_s$ and both only depend on the normal force and neither on the velocity nor on the area of contact (*Coulomb law*). No doubt, this picture is idealized and an entire discipline, called tribology, has evolved to study solid friction in depth³⁶. For our purpose it is, however, more convenient to concentrate only on the basic mechanisms because we are interested in explaining the generic effects of granular rheology without entering into material-dependent details. Friction has a crucial consequence on the level of the grains, namely that the system does *not* conserve energy as opposed to what happens on the molecular level. Another source of dissipation can be plastic deformation of grains due to the normal force acting at collisions. Again it seems most important that dissipation does occur and the complications arising from the non-linearities of plasticity seems less relevant. We will therefore in the following assume very simple dissipation laws. In fact more complicated laws have been used increasing the number of parameters without giving qualitatively different answers.

Let us consider a system of N spherical particles of equal density and with diameters d chosen randomly from a homogeneous distribution of width w around d_0 . When two particles i and j overlap (i.e. when their distance is smaller than the sum of their radii) three forces act on particle i :

- 1.) an elastic restoration force

$$\vec{f}_{el}^{(i)} = k_n m_i \left(\frac{1}{2}(d_i + d_j) - |\vec{r}_{ij}| \right)^\beta \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|}, \quad (1a)$$

where k_n is the elastic modulus (normalized by the mass), m_i the mass of particle i and \vec{r}_{ij} points from particle i to j . For Hooke's law $\beta = 1$ but in the case of spheres in three dimensions one has to choose $\beta = 3/2$ ³⁷ and for conical contacts one should take $\beta = 2$.

2.) a dissipation due to the inelasticity of the collision

$$\vec{f}_{diss}^{(i)} = -\gamma_n m_i (\vec{v}_{ij} \cdot \vec{r}_{ij}) \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|^2} = -\gamma_n m_i v_{ij}^n \quad , \quad (1b)$$

where γ_n is a phenomenological dissipation coefficient and $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$ the relative velocity;

3.) a shear friction force which in its simplest form can be chosen as

$$\vec{f}_{shear}^{(i)} = -\gamma_s m_i (\vec{v}_{ij} \cdot \vec{t}_{ij}) \frac{\vec{t}_{ij}}{|\vec{r}_{ij}|^2} = -\gamma_s m_i v_{ij}^t \quad , \quad (2a)$$

where γ_s is the shear friction coefficient and \vec{t}_{ij} is the vector \vec{r}_{ij} rotated by 90° . Eq. 2a is a rather simplistic description of shear friction. In many applications (arching, heap formation) it is important to include real static friction³⁸ which can be done by a static friction force²³. When two particles start to touch each other, one puts a "virtual" spring between the contact points of the two particles. If δs is the total shear displacement of this spring during the contact the restoring frictional force is $k_s \delta s$ (static friction). The maximum value of the restoring force is then according to Coulomb's criterion proportional to the normal force F_n and the proportionality constant is the friction coefficient μ . Cast into a formula this gives a friction force

$$\vec{f}_{friction}^{(i)} = -\text{sign}(\delta s) \min(k_s \delta s, \mu F_n) \quad . \quad (2b)$$

where δs is the shear displacement integrated over the entire collision time. When particles are no longer in contact with each other the spring is removed. Main source of static friction is the geometrical roughness of the surfaces³⁹ and the same effects of particle stopping can be obtained also without Eq. (2b) by using particles of complicated shapes, like crosses or polygons^{40,41}.

It is not straightforward to implement the above technique when the particles are allowed to rotate, i.e. able to roll on each other. In fact, when particles have strong deviations from the spherical shape rotations are suppressed. Often it is however useful to go an intermediate way and to include dynamic friction but not static friction and allow for the particles to rotate^{8,30,32}. In that case one uses a combination of Eqs. 2a and 2b:

$$\vec{f}_{dyn}^{(i)} = -\min(\gamma_s v_{ij}^t, \mu F_n) \quad (2c)$$

and also introduces equations of motion for the angular momentum of the particles.

When a particle collides with a wall the same forces act as if it would have encountered another particle of diameter d_0 at the collision point. One force that acts on all particles pulling them down is gravity, $g \approx -10\text{m/s}^2$. In most simulations presented here we use a fifth order predictor-corrector MD with $10^3 - 10^5$ iteration steps per cycle depending on the largest velocities appearing in the application. This algorithm vectorizes on the CRAY YMP, running at about $10 \mu\text{sec}$ per particle-update for $N = 200$. The programs were also run on 8 or 16 processors of an Intel iPSC/860 or on an IBM RS/6000. For massive parallel systems with distributed memory, MPI (Message Passing Interface) offers a convenient way to implement parallel programs. On the CRAY T3E-900 it takes about $50 \mu\text{s}$ per

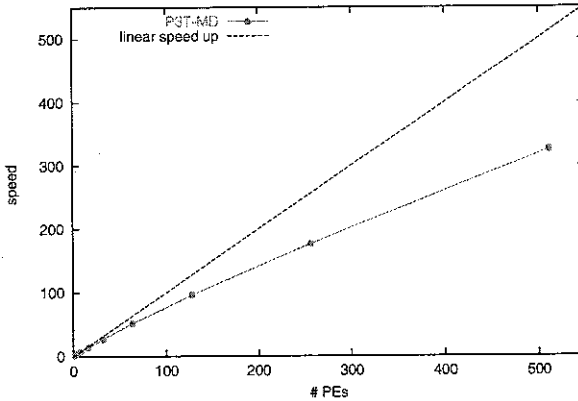


Figure 1. Fig. The speed-up of the molecular dynamics program is shown, i.e. the inverse execution time versus the number of processors used to parallelise the problems. The lower curve is measured and the upper curve shows the upper theoretical limit.

particle update on a single CPU. The program shows good speed-up to 512 CPU's (Fig. 1). In a metacomputing testbed it was running on 1258 CPU's coupling two T3E's at NIC/ZAM Jülich with a T3E at HLRS, Stuttgart. In a transatlantic metacomputing effort the same program set a world record for the largest particle number with 1,399,440,000 particles⁴².

3 Angle of Repose and Angle of Marginal Stability

As opposed to fluids the surface of granular media at rest is usually not flat but can show undulations (ripples, dunes) or form mountains (sand piles, heaps). This is due to the fact that up to a maximal angle, the so called angle of maximal stability θ_m , grains can arrange in a statically stable way. The angle of a heap of granular material can fluctuate between θ_m and a dynamic angle θ_r . Various experiments can be devised to obtain these angles.

Numerical studies have been made to investigate the angles θ_r and θ_m in two dimensions³⁹. They start by putting N particles randomly in a box and wait sufficiently long until under the action of gravity they have all settled on top of each other and come to rest. Next one side wall is removed. The particles violently fall out forming a big avalanche and after some time only a few remain inside the box. The pile that is left forms a certain angle which is what is called the angle of repose θ_r .

In principle the static friction coefficient μ defined in Eq. 2c should determine the slope of the pile. Since the angle of repose can easily be determined experimentally it is of interest to find the relation between θ_r and μ . This relation is essentially linear for $\mu < 0.2$. Recently⁴³ in fact these calculations have been extended to

larger values of μ and a curvature was observed which is close but not identical to the theoretical relation $\mu = \tan\theta_r$. For the range of parameters which interest in practice, however, the linear relation is a very good fit.

Molecular dynamic simulations for granular media have been performed for nearly all technologically relevant configurations: silos, pipe flow, vibrated beds, deformation of soils, segregation, rotating drums, heap formation and so on. Let us just cite two review articles of those applications ^{44,45}.

4 Direct Simulation Monte Carlo (DSMC)

The most time consuming part of a MD simulation is the calculation of the forces between the particles. For hard spheres the so called hard sphere MD or event driven (ED) algorithm offers a possibility to avoid this calculation. In contrast to MD it is however not well suited for parallelization. A method that avoids calculating forces and is very well suited to run on parallel computers is DSMC. This method was first proposed by Bird ⁴⁶ for the simulation of rarefied gas flows, recently it was also applied to dry granular media ^{47,48}.

One of the basic assumptions of DSMC is that the movement of the particles can be decoupled from their interaction. The system is integrated in time steps τ . At each time step every particle is first moved, according to the equation of motion, without interaction with other particles. External forces, such as gravitation, are taken into account here. To calculate the movement of the particles one can either use an analytical solution of the equation of motion or apply a standard numerical integration scheme to solve it. In this respect this method is less restricted than event driven (ED) simulations, where an analytical solution is required for a fast calculation of the evolution of the system.

Next, we take the particle-particle interactions, *i.e.* the collisions into account. In contrast to ED simulations, the exact times and places of these collisions are not calculated, but a stochastic algorithm is applied as described in the following:

The particles are sorted into spatial cells of linear size L and volume $V_c = L^d$, where d is the dimensionality of the system. Collisions occur only between the particles in the same cell, which ensures that only particles which are close to each other may collide. In every cell with more than one particle, we choose randomly

$$M_c = \frac{N_c(N_c - 1)\sigma v_{max}\tau}{2V_c} \quad (1)$$

pairs of particles. Here, N_c is the number of particles in the cell, σ the scattering cross section (for spherical particles, $\sigma_{2D} = 4R$, $\sigma_{3D} = 4\pi R^2$) and v_{max} is an upper limit for the relative velocity between the particles. In order to determine the correct number of collisions, we apply an acceptance-rejection method: For a pair of particles i and j the collision is performed if $\frac{|\vec{v}_i - \vec{v}_j|}{v_{max}} < Z$, where Z is independent uniformly distributed in the interval $[0, 1]$. This method leads to a collision probability proportional to the relative velocity of the particles.

Since the collision takes place regardless of the particle positions in the cell, we have to choose an impact parameter b in order to calculate the post collision velocities. Molecular chaos is assumed here; b is drawn from a uniform distribution

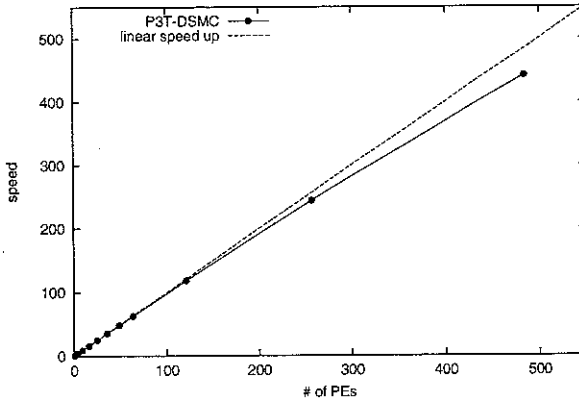


Figure 2. The speed-up of the DSMC program is shown, i.e. the inverse execution time versus the number of processors. The lower curve is measured and the upper curve shows the upper theoretical limit. Measurements have been performed on a CRAY T3E with a two dimensional system with $N = 500000$ particles and volume fraction $\nu = 0.1$.

in the interval $[-2R, 2R]$ in 2D or in a circle with radius $2R$ in 3D. The post collision velocities are now calculated as if the two particles collided with that impact parameter, i.e. like in event driven simulations.

Finally, the dissipation can be introduced by changing the normal component of the post collision velocity to $\bar{u}^{(n)} = -e\bar{v}^{(n)}$, whereas the tangential component remains unchanged.

So far only ideal gas properties have been considered; the particles have a scattering cross section but no real volume. One consequence of the excluded volume is an increase in the number of collisions. Several approaches exist to take this into account. One possibility is to multiply the right hand side of Eq. (1) with the Enskog factor χ . We take a different approach by replacing V_c in Eq. (1) with an effective free volume $\tilde{V} = V_c - V_b/\nu_{rcp}$, where V_b is the volume of the beads in that cell and ν_{rcp} is the packing fraction of a random close packing.

Following a proposal by F. J. Alexander et al.⁴⁹ we also introduce an additional advection process after a collision: each of the two colliding particles is moved by $2R$ into the direction of its momentum change.

Like every simulation method, DSMC is based on certain approximations. One is that the interaction between the particles can be modeled by collisions. Furthermore, neither the location nor the time of a collision is calculated exactly. To keep the error small, three conditions must be fulfilled: (i) the system should be in the collisional regime, (ii) the mean free path should be larger than the cell size, (iii) the mean time between two collisions should be larger than the time step. In particular these limitations restrict the validity of the method to relatively low densities.

4.1 Computational properties

Because the interaction of the particles is of short range, domain decomposition is the method of choice. Unlike in MD simulations the particles interact only with particles of the same cell, therefore no shadow rows have to be communicated. The reduced communication demand results in a better speed up (see Fig. 2).

A second advantage of DSMC is the low memory demand. Only the position and velocity of every particle have to be stored. On two coupled CRAY T3Es (one at HLRS/Stuttgart the second at PSC/Pittsburgh) we were able to simulate 1,759,165,695 particles of a granular gas.

5 Simulation of Particles in Fluids

A very important application of molecular dynamics concerns particles suspended in fluids. In that case, one must also simulate the velocity field of the fluid and compute the stress transfer of the fluid onto the particles and vice versa. The problem is very difficult since it comprises moving boundaries in the Navier-Stokes equations. These arise, because the fluid velocity must match the particle velocity on the particle surfaces. The fluid equations are discretized on a regular grid by placing the velocity vector on the vertices and the pressure values into the center of the cell (MAC) (cf., e.g. Refs. [51,53]). The equations are solved using an operator splitting technique implemented with a multigrid solver for the pressure equation.

Various attempts have been made to calculate the stress transfer from the fluid to the particle in an efficient way. One technique consists of expanding the velocity and pressure fields in polynomials around each particle⁵². Another technique consists of calculating the velocity gradient on the particle surface using linear interpolations of the values on the grid points close to the surface⁵⁴. Finally, a third technique is the marker method^{55,56,57} that is based on Ref. [50]. It consists in having on the surfaces and in the interior of each particle marker points (more than 30 points for three dimensional spheres) and convecting these marker points with the fluid. Springs connect the markers with their initial positions on the particles and mediate the force between fluid and particles. All these three techniques have been implemented in three dimensions on parallel computers. The marker technique is the most efficient one to simulate large systems and complex particle shapes. Over one million particles have been simulated using this method setting up the world record for particles in the fluid⁵⁸.

These methods have been used to simulate fluidized beds, segregation, aggregation under shear, etc. The method has also been tested on very careful experiments concerning velocity fluctuations and its size dependence for sedimentation⁵⁷.

References

1. A. Hansen and D. Bideau (eds.) *Disorder and Granular Media* (North-Holland, Amsterdam, 1992).
2. H.M. Jaeger and S.R. Nagel, *Science* **255**, 1523 (1992) and S.R. Nagel, *Rev. Mod. Phys.* **64**, 321 (1992).

3. C.S. Campbell, *Annu. Rev. Fluid Mech.* **22**, 57 (1990).
4. A. Mehta, *Granular Matter* (Springer, Heidelberg, 1994).
5. J.C. Williams, *Powder Techn.* **15**, 245 (1976) and *Fuel Soc. J.* **14**, 29 (1963).
6. A. Rosato, K.J. Strandburg, F. Prinz and R.H. Swendsen, *Phys. Rev. Lett.* **58**, 1038 (1987) and *Powder Techn.* **49**, 59 (1986); P. Devillard, *J. Physique* **51**, 369 (1990).
7. R. Jullien, P. Meakin and A. Pavlovitch, *Phys. Rev. Lett.* **69**, 640 (1992); R. Jullien and P. Meakin, *Europhys. Lett.* **22**, 523 (1993).
8. P.K. Haff and B.T. Werner, *Powder Techn.* **48**, 239 (1986).
9. M. Faraday, *Phil. Trans. R. Soc. London* **52**, 299 (1831).
10. P. Evesque and J. Rajchenbach, *Phys. Rev. Lett.* **62**, 44 (1989); *C. R. Acad. Sci. Ser. 2*, **307**, 1 (1988) and **307**, 223 (1988); C. Laroche, S. Douady and S. Fauve, *J. de Physique* **50**, 699 (1989); P. Evesque, *J. Physique* **51**, 697 (1990); J. Rajchenbach, *Europhys. Lett.* **16**, 149 (1991); E. Clément, J. Duran and J. Rajchenbach, *Phys. Rev. Lett.* **69**, 1189 (1992).
11. J. Walker, *Sci. Am.* **247**, 167 (1982); F. Dinkelacker, A. Hübler and E. Lüscher, *Biol. Cybern.* **56**, 51 (1987).
12. G.W. Baxter, R.P. Behringer, T. Fagert and G.A. Johnson, *Phys. Rev. Lett.* **62**, 2825 (1989).
13. C.-h. Lui and S.R. Jaeger, *Phys. Rev. Lett.* **68**, 2301 (1992).
14. O. Reynolds, *Phil. Mag. Soc.* **20**, 469 (1885).
15. Y.M. Bashir and J.D. Goddard, *J. Rheol.* **35**, 849 (1991).
16. S.B. Savage, *J. Fluid Mech.* **92**, 53 (1979); G.M. Homsy, R. Jackson and J.R. Grace, *J. Fluid Mech.* **236**, 477 (1992); S.B. Savage and K. Hutter, *J. Fluid Mech.* **199**, 177 (1989).
17. H.J. Herrmann, *J. Physique II* **3**, 427 (1993).
18. J.T. Jenkins and S.B. Savage, *J. Fluid Mech.* **130**, 186 (1983).
19. S.F. Edwards and R.B.S. Oakeshott, *Physica A* **157**, 1080 (1989); S.F. Edwards, *J. Stat. Phys.* **62**, 889 (1991); A. Mehta and S.F. Edwards, *Physica A* **157**, 1091 (1989).
20. O. Walton, in *Two Phase Flow*, ed. M. Roco (Butterworth-Heinemann, Boston, 1992), chapter 25.
21. G. Ristow, *Ann. Rev. Comp. Phys.*, ed. D. Stauffer (World Sci. Publ. Co., Singapore, 1994).
22. P.A. Cundall, Report AD/A - 001 602, U.S. Nat. Tech. Information Service, (Springfield, Va, 1974).
23. P.A. Cundall and O.D.L. Strack, *Géotechnique* **29**, 47 (1979).
24. R.J. Bathurst and L. Rothenburg, *J. Appl. Mech.* **55**, 17 (1988).
25. Y. Zhang and P.A. Cundall, Proc. Symp. on the Mech. of Particulate Media, 10th National Congress on Applied Mechanics (Austin, Tex, (1986); J.M. Ting and B.T. Corkum, Proc. 6th Int. Conf. Num. Meth. in Geomech., Int. Commission for Num. Meth. in Geomech. **1**, 305 (1988); R. Dobry and T.T. Ng, Proc. 1st U.S. Conf. on Discrete Element Meth., Nat. Sci. Found. (Washington, DC, 1989); J.A. Issa and R.B. Nelson, idem; P.J. Hassan, thesis Univ. of Toronto (1990).
26. M.R. Kuhn and J.R. Mitchell, Proc. 1st U.S. Conf. on Discrete Element

- Meth., Nat. Sci. Found. (Washington, DC, 1989).
27. L.J. Lorig and B.H.G. Brady, ISRM British Geotech. Soc. (Cambridge, 1984), p.105.
 28. R. Barbosa and J. Ghaboussi, Proc. Workshop on Const. Laws for the Anal. of Fill Retention Struct., Canada Oil and Gas Adm. (Ottawa, 1987); M. Grabinsky, thesis Univ. of Toronto (1989); J.M. Ting, B.T. Corkum, C.R. Kauffman and C. Greco, J. of Geotech. Eng., ASCE **115**, 379 (1989).
 29. T. Pöschel and H.J. Herrmann, *Europhys. Lett.* **29**, 123 (1995).
 30. G. Ristow, *J. Physique I* **2**, 649 (1992) and *Int. J. Mod. Phys. C* **3**, 1281 (1992).
 31. D.C. Hong and J.A. McLennan, *Physica A* **187**, 159 (1992).
 32. C.S. Campbell and C.E. Brennen, *J. Fluid Mech.* **151**, 167 (1985); P.A. Thompson and G.S. Grest, *Phys. Rev. Lett.* **67**, 1751 (1991); D.M. Hanes and D.L. Inman, *J. Fluid Mech.* **150**, 357 (1985); O.R. Walton and R.L. Braun, *J. Rheol.* **30**, 949 (1986); M. Hakuno, K. Iwashita and Y. Uchida, Proc 1st U.S. Conf. on Discrete Element Meth., Nat. Sci. Found. (Washington, DC, 1989);
 33. T. Pöschel, *J. Physique II* **3**, 27 (1993).
 34. M.P. Allen and D.J. Tildesley, *Computer Simulation of Liquids*, Oxford University Press, Oxford, 1987.
 35. D. Tildesley, in *Computational Physics*, edited by R.D. Kenway and G.S. Pawley, NATO Advanced Study Institute, Edinburgh University Press, 1987.
 36. K.L. Johnson, *Contact Mechanics* (Cambridge Univ. Press, 1989).
 37. L.D. Landau and E.M. Lifshitz, *Elasticity Theory* (Pergamon Press, Oxford, 1975).
 38. J. Lee and H.J. Herrmann, *J. Phys. A* **26**, 373 (1993).
 39. T. Pöschel and H.J. Herrmann, *Physica A* **198**, 441 (1993).
 40. T. Pöschel and V. Buchholtz, *Phys. Rev. Lett.* **71**, 3963 (1993); V. Buchholtz and T. Pöschel, *Physica A* **202**, 390 (1994).
 41. G.A. Kohring, S. Melin, H. Puhl and H.J. Tillemans, and W. Vermöhlen. *Comp. Meth. in Appl. Mech. and Eng.* **124**, 3 (1995).
 42. Benutzerinformation des Rechenzentrums der Universität Stuttgart, **11**, 1997.
 43. F. Cantelaube, Y.L. Duparcmeur, D. Bideau and G.H. Ristow, *J. Phys. I France* **5**, 581-596 (1995).
 44. H.J. Herrmann, Simulating granular media on the computer, *3rd Granada Lectures in Computational Physics*, eds. P.L. Garrido and J. Marro, Springer, Heidelberg (1995), 67-114.
 45. H.J. Herrmann and S. Luding, *Continuum Mechanics and Thermodynamics* **10**, 189-231 (1998).
 46. G. A. Bird, *Molecular Dynamics and the Direct Simulation of Gas Flow*. Oxford Science Publications, Oxford, (1994).
 47. M. Müller, S. Luding, and H. J. Herrmann. *Simulations of vibrated granular media in 2D and 3D*. (World Scientific, Singapore, 1997).
 48. J. J. Brey, M. J. Ruiz-Montero, and D. Cubero, *Physical Review E* **54**, 3664-71, Oct. (1996).
 49. F.J. Alexander, A.L. Garcia, and B.J. Alder, *Physical Review Letters* **74** 26, 5212-5215 (1995).

50. A.L. Fogelson and Ch.S. Peskin, A fast numerical method for solving the three-dimensional Stokes equations in the presence of suspended particles, *J. Comp. Phys.* **79**, 50–69, (1988).
51. A.J. Chorin, *J. Comp. Phys.* **2**, 12-26, (1967).
52. B. Wachmann, W. Kalthoff, S. Schwarzer, and H.J. Herrmann, *Granular Matter* **1** (2), 75-82, (1998).
53. R. Peyret and Th. D. Taylor, *Computational Methods for Fluid Flow*, Springer Series in Computational Physics, (Springer, New York, Berlin, Heidelberg, 1988).
54. S. Schwarzer, *Phys. Rev. E* **52** 6, 6461–6475, (1995).
55. S. Schwarzer in *Physics of Dry Granular Media*, ed. J.-P. Hovi, S. Luding and H. Herrmann, Proc. NATO ASI Cargèse, Corsica, (Kluwer Academic, Dordrecht, 1998).
56. K. Höfler, Räumliche Simulation von Zweiphasenflüssen, Universität Stuttgart (1997).
57. B. Wachmann and S. Schwarzer, *Velocity Fluctuations in Three Dimensional Monodisperse Suspensions*, Proceedings of the 6th Annual Conference of the Computational Fluid Dynamics Society of Canada, Quebec City (1998).
58. B. Wachmann and S. Schwarzer, *I. Jour. of Modern Physics* **C9** 5, 759 (1998).