Internal avalanches and restructuring in granular media

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Abstract

We study the phenomenon of internal avalanching within the context of recently proposed “Tetris” lattice models for granular media. We consider a packing of particles subjected to two different dynamics. In the first case, we arrest the system at different instances during an “aging” dynamics during which the packing slowly compactifies under shaking, and study the distribution of internal avalanches at each of these instances. In the second case, we define a recycling dynamics under which the system reaches a steady state under continued avalanching. We study the distribution of avalanches in this steady state. In the former case we investigate numerically the effect of the density of the medium on the avalanche distribution. In the latter case we develop a mean-field theory to help understand the reciprocal effect of the avalanches on the medium. © 1999 Published by Elsevier Science B.V. All rights reserved.

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Understanding the physics of granular materials has incited a lot of interest [1]. Of particular interest is their response to external perturbations since they behave neither like liquids nor like solids in most instances. As a result, there have been several experiments performed for studying compaction, segregation, shearing, surface avalanching; and several numerical and analytical approaches to explain the observed phenomenology. While there does not exist as yet a global understanding of all the existing phenomenology, there have been several attempts to construct simple models which still retain some essential features of granular packings (for a recent introduction to the overall phenomenology see [1]).

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The model we have investigated is one such, a recently proposed lattice model for describing slow dynamical processes in granular media [2]. The basic ingredients of this model are the geometric constraints involved in packing particles of different shapes. Even with just this feature, this model is found to reproduce a number of experimentally observed phenomena such as slow relaxation in compaction [2], segregation [3], as well as aging [4].

Within the context of this model, we study the response of granular packings to perturbations such as taking out a particle from the base of the packing. The ensuing disturbance, big or small, we call an internal avalanche. We study the distribution of these events under different conditions and find interestingly that packings are typically fragile, resulting in long-ranged avalanche distributions. We attempt to understand the effect of the medium on the avalanches as well as the effect of repeated avalanching on the medium.

We briefly review the definitions and some basic properties of the Tetris models used in our simulations. The model is defined as a system of particles occupying the sites of a square lattice tilted by 45° with periodic boundary conditions in the horizontal direction and a rigid wall at the bottom. The particles can have different "shapes," occupying a single or several lattice sites; the only constraint imposed is that particles cannot overlap. In each of the cases we study, we consider packings of particles with two different shapes: rods with two kinds of orientations (different orientations on the lattice can be considered to be different shapes because the dynamics does not allow rotation), "T"-shaped particles with two kinds of orientations or "crosses" with arms of randomly distributed lengths in the framework of the random tetris model (RTM) [5].

The system is initialized by inserting the particles at the top of the system, one at a time, and letting them move down under gravity. The particles perform an oriented random walk on the lattice until they reach a stable position defined as a position from which they cannot fall any further because of other particles below them. The particles retain their orientations as they move, i.e., as already mentioned, they are not allowed to rotate.

On this initial state, we perform two different kinds of dynamics. In the first of these, the packing is "shaken" for a certain amount of time till a higher density is reached (we refer the reader to [2,6] for more details of this procedure). When the desired density is reached the dynamics is arrested and measurements made for the size of an avalanche at this density. An ensemble averaging is performed in order to get a distribution of avalanche sizes. Fig. 1a illustrates the avalanche distribution over a range of densities for a single system size ($L_x = 200$, $L_y = 200$). As evident the avalanche distribution is sensitive to the structural changes in the packing caused by the shaking.

To understand how the density of the packing may affect an avalanche, we consider the microscopics of an avalanching process. The removal of a particle at the bottom creates a void in the lattice. This potentially destabilizes the neighbouring grains above, one (or both) of which may fall down to fill this void if the geometry of the packing
Fig. 1. (a) \( P(s) \) vs. \( s \) for the densities \( \rho = 0.76 - 0.83 \); (b) the scaling plot \( f(s^*) \) vs. \( s^* \) where \( s^* = s(\rho - \rho_c)^{1/\sigma} \) and \( f(s^*) = (s^*)^{-1} F(s^*) \). The densities scaled are \( \rho = 0.81 - 0.83 \) of the data shown in (a). The scaling parameters are \( \tau = 1.5 \pm 0.1 \), \( 1/\sigma = 1.5 \pm 0.1 \) and \( \rho_c = 0.79 \pm 0.1 \).
allows the motion (i.e., if the orientation of the grain fits the local conformation). In this case, the net effect is that the void propagates one lattice step upwards (or multiplies on contact with a trapped void) destabilizing its neighbors in the layer above and so on. How effective this process is in causing the restructuring of the configuration depends on the precise structure of the packing which in turn depends on the density. The avalanche is hence similar to a birth–death process and the density plays a role in determining the probabilities of birth and death. This leads us to make the following hypothesis for the avalanche distribution, commonly made for branching processes (as in directed percolation or in many self-organized critical models), that the avalanche size distributions obtained at different densities is obtainable from one single scaling function such as

\[ P(s, \rho) = s^{-\tau} F(\rho - \rho_c)^{1/\sigma}, \]

where \( \rho_c \) represents the location of the critical point and \( P(s, \rho) \) is the probability for avalanches propagating in a medium of density \( \rho \). If we make this hypothesis of a single critical density for the data, then the results for scaling the avalanche data in Fig. 1a is shown in Fig. 1b. The best value of the exponents \( \tau \) and \( \sigma \) are indicated in the figure.

In order to test this hypothesis of a single critical density in a simpler situation as well as elucidate the possible meaning of a critical density, we have also looked at a toy model which is a simple limiting case of the more general situation [6]. The Tetris model with rods of two orientations has a very simple ground state (highest density state) – the completely antiferromagnetic one. We take advantage of this fact by constructing the toy model in the following way. We begin with a \( \rho = 1 \) completely antiferromagnetic state and generate lower density states by randomly removing particles. After each removal, the system is allowed to re-settle into a stable state via the avalanche dynamics already described. The density here is just the total number of particles in the system. As a further simplification, we consider periodic boundary conditions in both \( X \) and \( Y \) directions. This allows us to eliminate system size effects as well as edge effects on the avalanche statistics. We call this example the fully periodic single domain (FPSD) model.

On this simplified version of the model, we perform the same set of measurements described earlier, in order to measure the avalanche distribution as a function of the density. The numerical results for avalanche size distribution as a function of the density are shown in Fig. 2a. In this case too the scaling hypothesis is satisfied and all the curves collapse for the indicated values of the parameters. (see Fig. 2b). As can be seen, the values of the scaling exponents are similar to those obtained for the original data. Further, the critical density here is just the least stable density of the packing.

From the above simulations, we conclude that the density dependence of the avalanche distribution is given reasonably well by Eq. (1). We now ask the reciprocal question to the one addressed so far, i.e. how does the avalanche affect the density, or more generally, the packing? To answer this, we consider the following dynamics for the evolution of the system. A particle is removed from a random position at the base as
Fig. 2. (a) $P(s)$ vs. $s$ for the densities mentioned for the FPSD model; the scaling plot with the scaling parameters $\tau = 1.45 \pm 0.05$, $1/\sigma = 1.5 \pm 0.1$ and $\rho_c = 0.76 \pm 0.01$. 
Fig. 3. (a) Typical avalanches (shaded particles) in the steady state for a system of T-shaped particles. The steady state shows characteristic ordering effects with the avalanches propagating preferentially in the boundary between two ordered regions; (b) \( P(s) \) vs \( s \) in the steady state for “T-shaped” particles and the “crosses” (RTM); \( \tau = 1.5 \pm 0.05 \) in both cases. The system sizes shown are \( Lx = 100, Ly = 500 \) and \( Lx = 200, Ly = 650, 1000, \) respectively, for the “Ts” and \( Lx = 100, Ly = 150, Lx = 200, Ly = 300 \) for the “crosses”. 

Before. Once the avalanche caused by this removal comes to a halt, the removed particle is added back at a random position at the top of the system [7]. This is continued till the system reaches a steady state. A similar dynamics for another model has also been investigated in Ref. [8].

Under this dynamics, the packing reaches an inhomogeneous steady state (see Fig. 3a). For all the particle shapes studied, the steady state was always found to be similar in nature. Namely, beginning from an initial randomly mixed state, the packing always “segregates” under the dynamics so as to form ordered high density grains separated by grain boundaries at lower densities. All avalanches preferentially propagate inside the grain boundaries.

The size distribution of the avalanches here too decays like a power \( P(s) \sim s^{-\tau}. \) This was studied for the three different types of particles described above. Time averages were performed in the steady state over \( \sim 10^5 \) configurations in order to obtain good statistics (see Fig. 3b).

From our previous discussion, if the density of the medium is a single parameter describing the medium, then Eq. (1) describes the effect of density on avalanches. In the following, we develop a mean-field theory for the reverse process – the effect of avalanches on the density under the above steady state dynamics. As argued above the avalanche is like a branching process on a lattice. However, there is a feedback effect
in that the avalanche distribution can in its turn affect the density of the system: large avalanches that reach the top tend to compactify the system and small avalanches make the system looser.

We can make the above arguments more precise in the following way. Let $\rho_h(t)$ be the cumulative density of the system up to height $h$ at time $t$. Then the density of the system at time $t+1$ will be

$$\rho_h(t+1) - \rho_h(t) = -1/L^2 + \alpha(t) \times h^2/L^2,$$

(2)

where $L$ is the linear size (height) of the system. The first term on the RHS represents the effect of removing one particle. This is the sole contribution of avalanches which die before reaching the height $h$. As a result their net effect is to increase the number of voids by one in the bulk of the system and hence decrease the overall density. On the other hand, those avalanches which reach at least a height $h$ increase the density of the system by an amount equal to the number of voids which escape at $h$. This is equal to the width of the avalanche at $h$. If the avalanches are self-affine (as in the case studied here), i.e. an avalanche of height $h$ has a width of $h^\gamma$, then the density increase is precisely given by the second term. The coefficient $\alpha(t)$ is related to the probability that an avalanche reaches the surface of the system and it is 0 with probability $P_0 = \int_0^{s^*} P(s) \, ds$ and 1 with probability $1-P_0$. Here $s^* = h^{1+\gamma}$ is the avalanche size cut-off at height $h$. If the avalanches in the steady state are distributed by a power law, i.e. $P(s) \sim s^{-\tau}$, then the requirement that in the steady state the average density $\langle \rho \rangle = \text{const}$ gives us the relation $(1-P_0(h)) h^\tau = h^2/h^{1+\gamma(1-\tau)} = 1$. This leads to the following scaling relation:

$$\tau = 1 + \gamma/(1 + \gamma).$$

(3)

The values of $\tau$ and $\gamma$ in our simulations are consistent with this relation. A more complete and self-consistent description of the observed phenomenology can be obtained complementing Eq. (2) with an equation for the avalanche distribution $P(s)$ in terms of $\rho$, i.e. with an equation

$$F(s(t)) = F(\rho(t)),$$

(4)

where $F$ indicates a generic function of $\rho(t)$ (for example Eq. (1)). The two coupled equations (2) and (4) should then describe the evolution of the system to a steady state given by a critical density $\rho_c$ with an avalanche distribution decaying as a power law. We consider a detailed study of the coupled equations for different explicit forms of $F$ (for avalanches propagating on a Bethe lattice) elsewhere [9].

The scaling relation (3) always holds for systems with open boundary conditions provided that there is a compact bulk packing. This poses an upper limit to the exponent $\tau$. For non-fractal bulk packings with a smooth free surface, $\gamma$ cannot be larger than 1 and hence $\tau$ cannot be larger than 1.5. An important point to note is that Eq. (2) is not valid for systems with very loose packings. In this case, particles can fall large distances in the course of an avalanche and compactify the system far below, an effect not taken into account in the above equation. We refer the reader to [7] for more details on the different upper bound obtained for loose packings.
In summary, we have studied in detail the phenomenon of internal avalanching in a packing of particles, within the context of recently introduced lattice models for granular media. We explore both the effect of the medium on the avalanches as well as the effect of the avalanches on the medium. We also develop a mean-field theory in order to address these questions.

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