Spontaneous Density Fluctuations in Granular Flow and Traffic

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Abstract. It is known that spontaneous density waves appear in granular material flowing through pipes or hoppers. A similar phenomenon is known from traffic jams on highways. Using numerical simulations we show that several types of waves exist and find that the density fluctuations follow a power law spectrum. We also investigate one-dimensional traffic models. If positions and velocities are continuous variables the model shows self-organized criticality driven by the slowest car. Lattice gas and lattice Boltzmann models reproduce the experimentally observed effects. Density waves are spontaneously generated when the viscosity has a non-linear dependence on density or shear rate as it is the case in traffic or granular flow.

1 Introduction

Sand or powder exhibit many rather astonishing and scarcely understood phenomena [1], [2]. Well-known are the so-called "Brazil nut" segregation [3], [4] and heap formations that occur under vibrations [5], [6], [7]. A series of experiments [8], [9], [10], [11] have given evidence that under certain circumstances density patterns are generated inside the flowing medium. Density waves have been observed in flow through pipes [12] and down inclined planes [13]. Another ubiquitous phenomenon in granular media seems to be 1/f noise measured in hourglass flow [14]. Baxter et al. [15] observed power law decay in the frequency dependent forces that act on the wall of a hopper. power law noise has been described by the theoretical considerations of self-organized criticality [16].

I present the results of Molecular Dynamics simulations[17], [18] showing complex density patterns with a power law spectrum. Then I discuss a traffic jam model[19] showing similar behaviour. Finally I show how the effects are reproduced using a lattice gas with dissipation [20] and a lattice Boltzmann model [21].

2 Simulations of granular flow

A good way to solve the equations of motion of moving sand are Molecular Dynamics (MD) simulations of inelastic particles with static and dynamic friction. MD simulations [22] have already been applied to granular media
to model segregation [23], outflow from a hopper [24], shear flow [25], convection cells on vibrating plates [26], [27], avalanches on a sand pile [28] and others. In the following I will discuss simulations of pipe flow performed by T. Pöschel[17] and then G. Ristow’s work on density fluctuation in hoppers[18], both made in two dimensions.

Take a system of $N$ spherical particles of equal density and with diameters $d$ either equal $d_0 = 1$ mm. These particles are placed into a hopper having an opening angle $\theta$ and at the bottom an opening of diameter $D$. $\theta = 0$ corresponds to a pipe. In the case $\theta = 0$ we have a pipe. When two particles $i$ and $j$ overlap (i.e. when their distance is smaller than the sum of their radii) three forces act on particle $i$: 1.) an elastic restoration force

$$F_{\text{elastic}}^{(i)} = Y\left(\frac{r_{ij}}{|r_{ij}|} - \frac{1}{2}(d_i + d_j)\right)\frac{r_{ij}}{|r_{ij}|},$$ (1)

in which $Y$ is the Young modulus and $r_{ij}$ points from particle $i$ to $j$; 2.) a normal dissipation due to the inelasticity of the collision

$$F_{\text{dissipation}}^{(i)} = -\gamma m_{\text{eff}}(v_{ij} \cdot r_{ij})\frac{r_{ij}}{|r_{ij}|^2},$$ (2)

where $\gamma$ is a damping coefficient that can be calculated from the coefficient of restitution which is a material property, $v_{ij} = v_i - v_j$ is the relative velocity between the particles and $m_{\text{eff}} = m_i m_j/(m_i + m_j)$ is the effective mass; 3.) a shear friction force can be chosen as

$$F_{\text{shear}}^{(i)} = -\gamma_s m_{\text{eff}}(v_{ij} \cdot t_{ij})\frac{t_{ij}}{|r_{ij}|^2},$$ (3a)

in which $\gamma_s$ is a shear friction coefficient and $t_{ij} = (-r_{ij}^y, r_{ij}^x)$ is the vector $r_{ij}$ rotated by 90°. Eq. 3a is a rather simplified description of shear friction which is proportional to the relative velocity of the particles. In order to allow for situations of static blocking in a hopper due to arching it is important to include real static friction, i.e. which does not depend on the velocities but rather the relative angle of the surfaces [29]. When two particles start to touch each other, one puts a “virtual” spring between the contact points of the two particles. Be $\delta s$ the total shear displacement of this spring during the contact and $k_s \delta s$ the restoring frictional force (static friction). The maximum possible value of the restoring force is then, according to Coulomb’s criterion, proportional to the normal force $F_n$ multiplied by the friction coefficient $\mu$ [29]. This gives a friction force

$$F_{\text{friction}}^{(i)} = -\text{sign}(\delta s)\min(k_s m_{\text{eff}} \delta s, \mu F_n)$$ (3b)

where $\delta s$ is the shear displacement integrated over the entire collision time. When particles are no longer in contact with each other the spring is removed.

In a collision of a particle with a wall the same forces act as if it would have encountered at the collision point another particle of diameter $d_0$ with
infinite mass. In some cases the walls are in fact made out of small particles themselves and in order to introduce roughness on the wall these particles are chosen randomly from a distribution of two radii. The only external force acting on the system is gravity $g$.

We see in fig. 1 a space-time diagramm of the density inside a pipe with 600 particles and periodic boundary conditions. The particles initially have homogeneously randomly distributed initial positions and velocities (left pipe). After some time spontaneously various patterns appear in the density: On one hand one has very dark regions, nearly constant in time. Then one sees black diagonal stipes of constant velocity down the pipe. Finally there are also some lighter horizontal lines. We want to try in the following to explain these rather complex structures.

![Figure 1. A pipe plotted vertically at regular time steps next to each other. Time goes from left to right. Gravity acts from top to bottom. This picture was made by T. Pöschel. For more details see ref. 17.](image)

Strong density fluctuations have also been observed in hoppers of opening angles $\theta = 30^\circ$ for which the density at a position six particles diameters above the outlet has been measured as a function of time[18]. In fig. 2 we see the Fourier transformation of this density in a log-log plot. Clearly the data fall on a straight line over nearly two decades. The slope is about $\alpha = -1.35 \pm 0.1$ obtained by a least square fit. That means that we have a power law spectrum of the form $1/f^\alpha$. 
Figure 2. Power spectrum of the density fluctuations in a hopper[18].

3 Continuous traffic model in one dimension

We all know the seemingly erratic motions of cars jammed on highways. One wonders whether they are due to a random behaviour of the individual drivers or if there is an intrinsic chaotic mechanism. In favour of the first hypothesis is the existence of regular kinematic waves in dissipative systems with excluded volume [30]. For this reason many traffic models include rather important statistical noise in time [31]. In favour of the second hypothesis are measurements performed on Japanese highways showing a $1/f$ spectrum in the Fourier transformed density fluctuations [32] which might stem from some self-organized criticality [16]. It is therefore interesting to see if a traffic model without noise is able to give the observed erratic behaviour and its $1/f$ spectrum.

We consider a continuous one-dimensional model of ref. 19. The system has length $L$ with periodic boundary conditions. Velocity $v_i$ and position $x_i$ of a vehicle $i$ are continuous variables. The update rule is as follows:

For high velocities with respect to the gap the car slows down:

\[
\text{if } v > \Delta x - \alpha \quad \text{then} \quad v \rightarrow \max(0, \Delta x - 1); \quad (4a)
\]

(the "max" is only necessary to prevent negative velocities).

For small velocities with respect to the gap and for $v < v_{\text{max}}$ the car accelerates:

\[
\text{if } v < \Delta x - \beta \quad \text{and} \quad v < v_{\text{max}} \quad \text{then} \quad v \rightarrow v + \min(1, \gamma \times \Delta x). \quad (4b)
\]
Note that for $v_{max} = 5$ this rule allows maximum speeds up to nearly six and that $\beta > \alpha$.

After updating the velocity for all vehicles according to the rules of eq. 4, we move all vehicles simultaneously by a distance equal to their velocities.

In the simulations of ref. 19 $\alpha = 0.5$, $\beta = 3.0$, and $\gamma = 0.1$ were used. Initially $N = [\rho \cdot L]$ vehicles were placed on sites 1 to $N$, all with velocity zero, where $\rho$ was chosen small enough to prevent the last car hitting the first one through the periodic boundary conditions. Starting from this totally ordered initial state, the system was allowed to evolve according to the above rules. Only the speed of the first vehicle was kept fixed at $v_{lead} = 4.99999$ after its initial acceleration. This is a simplification of the well known situation where a number of fast vehicles follow a slower one which they cannot pass. The equally spaced cars rapidly evolve into a fluctuating state. In this new state density increases giving rise to very short bursts (traffic jams) of very different sizes which redistribute the cars backwards such that in some cases they even start again in equally spaced patterns.

![Graph: Power spectrum as function of frequency $\omega$ (unpublished result of K. Nagel)](image)

Figure 3. Power spectrum as function of frequency $\omega$ (unpublished result of K. Nagel).

This chaotic behaviour is an intrinsic consequence of the dynamics and not just the amplification of numerical round-off errors, as has been shown by comparing single precision with double precision calculations. The behavior of the model (i.e. the formation of the collective shocks) is robust with respect to parameter changes within at least the following range $0.1 \leq \alpha \leq 0.6$, $2.0 \leq \beta \leq 5.0$, $0.08 \leq \gamma \leq 0.12$, $4.5 \leq v_{lead} \leq 4.99999$.

We see on the right side of fig. 3 the power spectrum (square of the am-
plitude of the Fourier transformation) of the time sequence of the density measured at a fixed position of the highway. We see that the data follow a power law over more than two orders of magnitude with a slope of $1.4 \pm 0.1$. In addition one sees a peak at $\omega \approx 1/6$ probably due to the kinematic wave although its exact origin is not yet clear.

The traffic model presented here is reminiscent of the so-called train model for earthquake dynamics [33]. Instead of pulling at one end, the slower car may be seen as pushing against the other cars which want to move faster. This leads to a slowly increasing average density, and at some time this density locally exceeds a critical threshold. The reaction is a more or less drastic slowing down of the corresponding vehicle, which may or may not force the next vehicle to slow down as well. By this mechanism, avalanches of all sizes are generated, which may propagate through the entire traffic jam.

4 Lattice models in two dimensions

The flow of granular media can be described by the concepts of fluid mechanics. One must, however, consider that as opposed to classical fluids there is local dissipation of energy. Taking into account the dissipation rate in the energy balance equations it has been possible to predict the existence of an instability [34], [35]: Slightly denser regions have more dissipation and therefore lower pressure which itself generates a flow that will enhance the density. So, the dissipation will be responsible for the formation of clusters of high density and these have been observed [34]. It has also been possible to derive from this kinetic gas theory [35], [36], [37] that the viscosity increases very sharply with density.

One new way to solve the equations of motion of fluids are the so called Lattice Gas (LG) [38] and Lattice Boltzmann Models (LBM) [39]. These models are defined on a lattice with velocity vectors that can only point into few discrete directions and all have the same length. For the LBM this simplification is somewhat compensated by the fact that on each site one has more real degrees of freedom (six on a triangular lattice) than in the classical numerical techniques allowing for the definition of a local shear or a local rotation.

Let us describe the Lattice Boltzmann Model as used in ref. 20. We consider a triangular lattice, and on each site $x$ we have six real variables $N_i(x, t)$, $i = 1, .., 6$, representing (counted counter clockwise) the densities of the particles going in the direction $i$ of the lattice. (For convenience we will in the following omit the site index $x$ and denote by $N'_i$ the value of the particle density after collision.) One updating of the system ($t \rightarrow t + 1$) is given by the collision step at which the six $N_i$ are updated at each site through

$$N'_i = N_i + \lambda (N_i - N_i^{eq})$$

(5)

where $\lambda$ is a relaxation parameter and the propagation step at which each $N_i$ is shifted to the site of the nearest neighbor in direction $i$. Eq. (5) produces a
relaxation towards the equilibrium densities $N_i^{eq}$ which is numerically stable provided the relaxation constant fulfills $-2 < \lambda < 0$. The value of $\lambda$ sets the kinematic viscosity of the fluid. The equilibrium densities are given by

$$N_i^{eq} = \frac{\rho}{6} (1 + 2u \cdot c_i + 4(u \cdot c_i)^2 - 2u^2)$$

(6)

where $\rho$ is the mass density at site $x$

$$\rho = \sum_i N_i$$

(7)

where $c_i$ the unity vector along direction $i$ and $u$ the velocity vector at site $x$ defined through the momentum density per site

$$\rho u = \sum_i c_i N_i$$

(8)

The equilibrium distribution $N_i^{eq}$ given in Eq. (6), is chosen to give mass and momentum conservation in the collision step. The flow will be forced into the direction of gravity $g$, which is pointing parallel to the walls of the pipe. For that purpose an additional step is added after the collision step which is defined by $N_i'' = N_i' + \frac{1}{3} c_i \cdot (\rho g)$. Periodic boundary conditions are imposed in the direction of gravity in which the system has a length of $L_1$. In the perpendicular direction one has walls separated by $L_2$ lattice spacings. The lattice orientation is such that one of the lattice directions is parallel to the walls. At the beginning of the simulation the average density $\bar{\rho}$ is fixed. It is an important parameter of the model which because of mass conservation stays constant in time. We initialize the system by having the same values of the $N_i$ on each site and then let the system evolve to its steady state. In the case of the stable flows steady state is reached after 2000 or 3000 time steps. In the case of the unstable flows that develop density waves, the simulations might take up to 20000 time steps to reach steady state. At the walls no-slip conditions were used.

The parameter $\lambda$ depends on the kinematic viscosity $\nu$ like [38] $\lambda = -\frac{1}{3}(0.25 + 2\nu)^{-1}$. We will consider that $\nu$ is a function of the local density $\rho$. One dependence that generates the density waves in granular media is to assume that the viscosity depends on density. Within the kinetic gas theory of granular media[35], [36], [37] the relation $\nu \propto (\rho - \rho_c)^{1/3}$ has been derived. Since the above relation imposes a maximum density $\rho_c$ it is rather difficult to implement it directly within the context of the LBM where the particles do not have an exclusive volume. We therefore choose a piecewise linear relation of the form $\nu = \nu_{min}$ if $\rho \leq \rho_t$ and $\nu = \nu_0 + \gamma(\rho - \bar{\rho})$ for $\rho > \rho_t$ (see left side of fig. 4). $\bar{\rho}$ is the average density and the threshold density $\rho_t$ is chosen to make $\nu$ a positive continuous function of the density.
Figure 4. Left side: density dependence of the viscosity chosen in the simulations. Right side: density in the center of the channel as a function of the position $X$ along the channel for $\rho_1 = 2.962$, $\bar{\rho} = 3.0$, $g = 3.33 \times 10^{-5}$ and $L_2 = 64$. The curve of crosses is for $L_1 = 256$ and 60,000 iteration steps after the initial perturbation. The other curves correspond to $L_1 = 512$ and 5000 (thick line), 60,000 (full line) and 60,025 (dashed line) iterations after the perturbation was applied. The slope $\gamma = 6.25$ and the minimum viscosity $\nu_{\text{min}} = 0.01$. (from ref. 20)

To trigger the density waves it is necessary to introduce a small perturbation. We did this by making a 0.3% relative density difference. This perturbation was performed by introducing a small amount of momentum on one line across the pipe, keeping the mass unchanged. On the right side of fig. 4 we see that this initially very weak perturbation dramatically builds up and develops into a density wave of over 10% density contrast. For a pipe of same width but half the length, i.e. a different aspect ratio the wave is similar but has a less pronounced profile.

We checked that the complex shape of the waves does not reflect the detailed way in which they were initiated by triggering the density wave using two spatially separated perturbations, rather than just a single one. There seems to be no characteristic wavelength: since fig. 4 shows that the waves have roughly the same shape on the scale of the channel length and only the amplitude depends on the system size.

One can also formulate a lattice gas model\[38\] such as to include local dissipation of energy by introducing rest particles and special collision rules. Various variants of such a dissipative lattice gas have been introduced recently and used to simulate pipe flow \[21\], \[40\], \[41\], settling of particles under gravity \[42\] and the formation of heaps \[43\]. Particles live on a triangular lattice
where they can be either moving on a bond or at rest. At each time step the moving particles are propagated by one lattice unit in their direction of motion. When particles collide their character can be altered when one of the collision rules shown in fig. 5 applies. These rules create rest particles with probability \( p \) which describes dissipation. Parameters \( g \) and \( b \) are analogously introduced to describe the action of gravity and the roughness of the wall respectively.

Figure 5. The collision rules for the dissipative lattice gas (from ref. 43)

The LG model has been applied to a pipe with periodic boundary conditions and \( g \) acting along the pipe. In fig. 6 we see the result for the power spectrum of the time evolution of the density in pipe flow. On top of a \( 1/f^\alpha \) spectrum with \( \alpha = 1.33 \pm 0.02 \) we see a peak which corresponds to kinematic waves that travel around due to the periodic boundary conditions. Therefore its position depends on the length of the pipe. Recently also a dimensional
argument has been put forward [44] to argue that $\alpha = 4/3$. In the more realistic situation of open boundary conditions a power law spectrum is only observed at a specific density [40] which in fact corresponds to the maximum flux.

![Graph showing power law spectrum](image)

Figure 6. Log-log plot of the Fourier transformation of the density in a pipe of length 220 and width 11. The straight line is a least square fit with slope $-1.33 \pm 0.02$ (from ref. 21).

5 Conclusion

We reported that complicated density patterns are generated when granular material flows through a pipe or a hopper. In Molecular Dynamics simulations two ingredients were found[18] essential to generate them: static friction and a large enough surface roughness of the walls. The static friction tends to align the particles, i.e. to form fronts of particles moving exactly with the same vertical velocity. These fronts are nucleated randomly at the walls and their size distribution (density contrast) goes by itself into a critical state namely a power law distribution. It therefore has the properties of self-organized criticality (SOC) [16]. Traffic jam models show[19] similar behaviour due the same effect, namely local dissipation of energy. In a fluid picture, a lattice Boltzmann model shows[20] that the density dependence of the viscosity that one encounters in granular media generates waves of low density. A lattice gas model with local dissipation gives both, kinematic waves and SOC in the power spectrum.

The following picture emerges: Due to the inelastic collisions between particles (also cars) an instability[34], [35] tends to form clusters of high density (dark horizontal stripes in fig. 1). These clusters self-organize into a critical state giving a power law spectrum. As in classical traffic models the density dependence of the flux generates kinematic waves (dark tilted lines in fig. 1). The density dependence of the viscosity as predicted by kinetic gas theory builds up waves of low density (light horizontal lines in fig. 1). We see that introducing dissipation to a gas of particles is enough to produce many interesting phenomena, that occur in granular materials. Experimentally this
picture has only partly been confirmed. Some authors do find the 1/f spectrum [12], [14] others do not [13]. More systematic work in this direction is necessary.

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