Simple Models for Fragmentation

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Abstract

We study an iterative stochastic process as a model for two dimensional discrete fragmentation. This simple model, which fulfills mass conservation, allows us to introduce different microscopic stress configurations. The introduction of elementary size pieces that can not be broken further imposes a limit to the fragmentation process. Despite their simplicity, our models present complex features that reproduce some of the experimental results that have been obtained previously. Different fragment size distributions are obtained depending on the breaking rules of our models. For some regimes a power law behavior is obtained. For this reason we propose them as basic models that can be substantially refined to describe the fragmentation process of rocks.

1 Introduction

In this work we study simple models for fragmentation processes defined as discrete time stochastic processes on a two dimensional square lattice. At each time step of the process only one piece of the material is broken, chosen according to the rule of the model. We are interested in determining the fragment size distribution.

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Fragmentation processes are common phenomena in nature, that appear in a wide range of scales and situations. In [1] Turcotte gave a long enumeration of natural fragment distributions (asteroids, coal heaps, moraines, etc) for which power-law distributions were measured with exponents ranging from 1.9 to 2.6 concentrating around 2.4.

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In some way, the power law behavior for small fragment masses or sizes seems to be a universal characteristic of the instantaneous breaking of brittle objects.

There are careful experiments in one dimension, see [2], in which long thin glass rods are broken by vertically dropping them. Depending on the height from which the glass rods are dropped, the fragment size distribution varies from a log-normal shape for smaller heights to a power law for increasing height. Stochastic models for one dimensional fragmentation can be found in [3], where randomly fracture points between 0 and 1 are chosen. The fragment size distribution is completely determined by the a priori random distribution of the fracture points. Power-law, exponential and log-normal fragment size distributions can be obtained. By the introduction of a minimal fragment size (a piece that can not be broken further) and a Poisson distribution for the number of fragments into which pieces are broken, the power law behavior is obtained at some stages of the fragmentation process.

In [4], this power law behavior for the fragment size density distribution \( f \) is given in the limit of small fragment masses, by the relation:

\[
  f(x) \propto x^{-\left(2 - \alpha/3\right)}
\]  

(1)
where \( z \) is the mass (or the volume) and the exponent \( \alpha \) varies between 0.5 and 1.0 for instantaneous fragmentation and highly energetic breaking. In general, \( \alpha \) depends on the material, the kind of fracture process and the external load.

In ref. [5], a model for the fragmentation of gas clouds via gravitational condensation is developed, under the assumptions of mass conservation and no presence of pressure. Initially, clouds split (condense) into \( q \geq 2 \) equal mass clouds. The process continues in a self-similar way such that the dynamic equations can be solved analytically. Further assumptions on the model allow to prove that the fragment mass distribution follows a power law behavior in the steady state.

The fracture process of a single grain has been studied by many authors. For instance, Gilvary [6] obtained its results based on the idea of pre-existing flaws (cracks) randomly distributed in the material, that are activated and propagate until they meet another flaw.

A mean-field type approach to describe the fragmentation process can be formulated through the concentration \( c(z,t) \) of fragments of mass lower than \( z \) in time \( t \) by the rate equations [4]:

\[
\frac{\partial c(z,t)}{\partial t} = -a(z)c(z,t) + \int_{z}^{\infty} c(y,t)a(y)f(z/y)dy
\]  

(2)

where \( a(z) \) is the rate at which fragments of mass \( z \) break into smaller ones (this quantity is supposed not to depend on time) and \( f(z/y) \) is the conditional probability that a fragment of mass \( z \) was produced from a fragment of mass \( y \geq z \). Using scaling and homogeneity assumptions, some exact results are obtained in Ref. [4], if some assumptions are made about \( f(z/y) \). In general the solutions of the former equations are however very complicated to obtain.

Stochastic processes discrete in space and time have also been studied as
models for fragmentation phenomena using cellular automata. In Ref. [7] two and three dimensional cellular automata are proposed to model the power law distribution resulting from shear experiments of a layer of uniformly sized fragments. The fracture probability of a fragment is determined by the relative size of its neighbors. A larger probability is obtained for fragments with a larger number of equal size neighbors. If two blocks have the same fracture probability a random decision is made. The process continues until no blocks of equal size are neighbors. In all cases a power law size distribution is obtained, with an exponent that depends on the former parameters.

In this paper we propose simple discrete stochastic processes on a two dimensional square lattice of size $2^n$. At each step only one piece of the material is broken into two equal size fragments (this is the reason for using system sizes of $2^n$). The breaking follows specific criteria based on microscopic models of forces, causing either horizontal or vertical breaking. The iterative breaking process stops when a piece of the smallest possible size (unit length) is forced to break. How we choose the piece to break and how the fragmentation process continues will be defined in the next section.

Our objective in this work is to measure the fragment size distribution for our models, and investigate under what circumstances these simple models present power law behavior in the limit of small fragments.

2 The Model

Each realization of our stochastic discrete process for modeling fragmentation considers the breaking of only one piece of the material assuming area (mass) conservation. As initial situation, we consider a two dimensional square lattice of size $2^n$. This initial piece of material can be broken either in the $x$ or in the $y$ axis direction (horizontal or vertical fracture) into two new pieces of equal area. In each corner of the piece we put independently and uniformly distributed random numbers between 0 and 1. These numbers
should represent the initial (scalar) forces acting on the material (one could also consider them to be local displacements), see figure 1.

For each step of our stochastic breaking process the following ingredients must be defined:

a) A definition and interpretation for the scalar forces as functions of the random numbers in the corners.

For this purpose, we will consider two classes of forces: traction or compression and shear forces, that can be computed from the random numbers in both directions by:

traction or compression:

\[ f_x = |a - d| + |b - c|, \quad f_y = |a - b| + |c - d| \quad (3) \]

shear:

\[ f_x = |(a - d) - (b - c)|, \quad f_y = |(a - b) - (c - d)| \quad (4) \]

The modulus is considered for both kinds of forces because we are interested in the total forces acting on the material. The forces are supposed to follow a one dimensional Hooke’s law.

b) A criterium to choose the piece of material to be broken.

Based on the above definitions for forces at each iteration of the stochastic process the more stressed piece of the material is broken. The piece with maximum stress among both axis directions is selected. Specifically, a piece \( P_1 \) with forces given by \( f_x^1 \) and \( f_y^1 \) is more stressed than a piece \( P_2 \) with forces \( f_x^2 \) and \( f_y^2 \) if:

\[ \max(f_x^1, f_y^1) \geq \max(f_x^2, f_y^2) \quad (5) \]
which clearly defines an order relation between the pieces of the material depending on the definition of the forces.

c) A criterium to choose the orientation of the fracture in the selected piece.

Once the piece to be broken is selected by applying the criterium defined in b), the fracture orientation must be defined. In this case the decision is straightforward: an horizontal cut is made if \( f_y \geq f_z \), otherwise a vertical cut is made. Two new pieces of the material of equal area are generated.

d) A criterium to define how the breaking process continues.

The idea of our model is to continue the fragmentation process in a self-similar way. In order to do that we have to define the new scalar forces acting on each piece. The external forces of the fractured piece are maintained as it can be see in figure 2, in which the two new pieces are represented after a vertical break. But new forces, say \( a', b', c', d' \), must be defined. We suppose Newton's law of action = reaction:

\[
\text{horizontal cut} \quad b' = a' \quad \text{and} \quad d' = c' \quad (6)
\]

\[
\text{vertical cut} \quad d' = a' \quad \text{and} \quad c' = b' \quad (7)
\]

With this assumption, only two new forces must be defined to continue the fragmentation process. We will consider two possible ways to choose these numbers. The first one is to choose independently random numbers between the corresponding values of the external forces, i.e.:

\[
a' = b' = \lambda a + (1 - \lambda)b \quad \text{and} \quad c' = d' = \lambda' d + (1 - \lambda')c \quad (8)
\]

where \( \lambda \) and \( \lambda' \) are uniformly distributed random numbers between 0 and 1. This kind of model will be called model A.

In the second case, the new forces are independently distributed random numbers between 0 and 1 without any dependence on the previous forces.
This kind of model will be called model B.

e) A stopping criterium for the fragmentation process.

At this point we have to consider the discrete nature of the model. This means that we cannot continue the breaking process for ever. We have to define stopping criteria, that can also be considered as typical for the evolution of the fracture process. It seems natural to impose that it is not possible to break a rectangular piece along a direction into which it has a length of unity.

We will consider two stopping criteria. In the first one we stop the global fragmentation process if the maximally stressed piece is tried to be broken along an elementary side. We call this criterium "fast stopping". The second one will be to continue the fragmentation process with the second most stressed piece and so on until all the pieces of the material have at least one side of length unity. We call this criterium "relaxed stopping".

Clearly, all the former considerations define different stochastic models of fragmentation in the sense that different fragment size distributions will be obtained, and this is exactly the purpose of this work: We will show that even these simple models of fragmentation, based on microscopic strength considerations and well defined (deterministic) breaking criteriums, can give interesting results that agree with the experimentally available data.

3 Results

In this section we present and discuss the numerical results obtained from the study of our models defined in the last section. We average our results over many (between 5.000 and 10.000) independent random initial conditions, characterized by the initial force configuration, i.e., the four random forces. As we said before, the system linear size is \(2^n\) where \(n\), for our simulations, varies between \(n = 64\) and \(n = 32768\). This wide range for the system size
will be explained later.

For each model we computed the fragment size (area) distribution as a function of time. The maximum number of steps the stochastic process can make is \(2^n - 1\). This gives an exponential bound for the evolution time that was observed for the models with relaxed stopping criterion. For the fast stopping criterion a transient time of order \(O(n)\) was observed.

Let us first consider the fast stopping criterion under traction (eq. (3)) for models A (fig. 3) and B (fig. 4). We see from these figures that the process does not have enough time to generate a large quantity of small area fragments. In the case of Model A, (figure 3), the number of big fragments is almost constant up to a certain value that depends on the original size of the system, a fact that can be explained from the definition of the model, because each time that a fragment breaks into two new pieces, the difference between the new forces on the generated fragments becomes smaller. Thus, the probability of generating fragments of small area is small. The time evolution in this case is very fast. Mostly, large area fragments are generated (about 75% of the fragments), which explains the fast convergence. On the other hand, Model B, (see figure 4), shows an almost log-normal behavior, due to the independent choice of the new fragments generated from the fracture process.

Next we consider the relaxed stopping criterion again for models A and B using eq. (3). The fragment size distribution is shown in figures very 6, respectively. The computations (made on a Sun 4 Sparc 2) were very time consuming due to the exponential time evolution for the relaxed stopping criterion. Model A, (figure 5), generates roughly a power law behavior for small area fragments. A large fraction of the fragments have the shape of long thin rods, which suggests a kind of shear effect inside the system. This feature of the model can be explained from the definition of the forces on the generated fragments: each time that a piece is broken its forces differences becomes smaller and thus the fragmentation process can be stopped at early
stages, generating large area pieces. By the other hand, the relaxed stopping
determines the shape of the fragments as thin rods. These two features can
produce the shear effect, but a more convincing explanation must be stated
from a theoretical study of the model. The fraction and the exponent of
the power law depend on the initial size of the material. For the systems
studied the exponent for small area fragments is equal to $-1.2 \pm 0.4$. Model
B generates a power law fragment size distribution, with an exponent which
increases with the size of the studied system. This result can be explained
as a finite size effect. Furthermore, due to the high requirements computer
time needed for the models with relaxed stopping, it is difficult to predict
or extrapolate an exponent for $n \to \infty$. Despite of this fact, we have found
power law behavior over the entire range of the area value, which is itself an
important result. For the system size of $n = 64$ the power law is of the form:

$$\log F(s) = \alpha \cdot \log(s) + \beta$$

(9)

where $F(s)$ is the fragment size distribution, $s$ is the area of the fragments and
$\alpha = -0.85\pm0.09$, $\beta = 1.66\pm0.03$. The former exponent does not agree with
the available three dimensional experimental data, which gives $\alpha = -1.6\pm0.2$
since our model is two-dimensional, but represents an important feature of
our model, because a power law behavior is produced by a simple stochastic
model.

Finally, we studied the relaxed stopping criterium with shear forces defined
by eq. (4). In this case, the fragments typically have the shapes of long
thin rods. For the system sizes studied, $n = 32, 64$ and $128$, the fragments
are mostly thin rods of size $1 \times 2^n$. The fact that the thin rods have an
elementary unit length is not surprising due to the definition of the relaxed
stopping criterium. But, the fact that the other side of the long thin rod is
substantially larger, needs more attention and simulations.
4 Conclusions

In this work simple models for two dimensional discrete fragmentation were studied. Different behaviors were obtained for the fragment size distribution which includes log-normal and power law behavior, depending on the definition of the parameters of our models, which are: the scalar forces acting on the fragments; the selection of the piece to break; the choice of the fracture orientation (vertical or horizontal); the definition of how the fragmentation process must continue and finally, the definition of a criterium to stop the fragmentation. The quasi-log-normal distribution can be explained from the random processes. The power law distribution is a non-trivial result.

The existence of elementary pieces that cannot be broken anymore introduces an arbitrary assumption. This limitation can be avoided by considering models that are discrete in time but not in space. We are actually developing such continous spatial models taking into account similar assumptions as in the discrete models studied here.

Acknowledgments

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References


Figures

Figure 1. Initial piece of material, where $a, b, c, d$ are independently and uniformly distributed random numbers.

Figure 2. Vertical Breaking the piece of figure 1.
Figure 3. Model A using eq. (3) and fast stopping criterium for $n = 32768$ and averaging over 100,000 samples.

Figure 4. Model B using eq. (3) and fast stopping criterium for $n = 32768$ and averaging over 100,000 samples.
Figure 5. Model A using eq. (3) and relaxed stopping criterion for $n = 64$ and averaging over 100,000 samples.

Figure 6. Model B using eq. (3) and relaxed stopping criterion for $n = 64$ and averaging over 100,000 samples.