On Oscillations in Cellular Automata.

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Abstract. – We investigate cellular automata where some global quantity varies periodically or quasi-periodically with time. We find that these systems are highly predictable, and can be rather well described by a set of mean-field variables. We conclude that this is not a collective phenomenon—where different subsystems are supposed to synchronize—but rather like many very weakly coupled oscillators fluctuating around one exact frequency. The global quantity measured is a mean taken over all these subsystems, and gets more precise the larger the system is.

The temporal behaviour of spatially extended systems of many degrees of freedom has recently attracted much interest. One question of special importance is under which circumstances a global quantity can show a non-transient evolution. It has been argued [1] that isotropic systems governed by local interactions only cannot exhibit non-trivial global behaviour. Despite these arguments, there are examples of cellular automata (CA) with periodic or quasi-periodic oscillations [2, 3]. Before we review the main features of these systems, we had better make clear what we are heading for. Any observable measured as an unweighted average over the whole system can be considered as global quantity. If this quantity varies over time, the conclusion is that spatially separated parts in the system have access to the same information. How can this be in a very large system? Some influence from the outside reaching all parts of the system at the same time is one explanation. If all interactions were local, one could instead think of a synchronizing mechanism making the cells cooperate and together perform a collective behaviour. The problem is that the information exchange would be too slow if the system were very large. Any kind of disturbances would thus create moving phase boundaries, which eventually would make the behaviour seize as shown in [1]. The only remaining solution we can think of, which is further exploited in this work, is that a spatially homogeneous initial configuration gives all regions of the system the same initial information. If a proper updating rule can be found, there is no need for communication between parts far away, because all parts will travel along the same trajectory in some phase space.

On a regular lattice of size $L^d$, an isotropic, deterministic, two-state CA will be defined by a totalistic rule, i.e. the state (0 or 1) of each cell is determined by the sum of all spins in the
nearest neighbourhood, the site itself included. All cells are updated synchronously, and we normally use periodic boundaries, although it does not affect the results. We consider here «rule 33» in three dimensions [3], which can be described as follows:

\[
\sigma_i(t + 1) = \begin{cases} 
1, & \text{if } h_i(t) = 0 \text{ or } h_i(t) = 5, \\
0, & \text{otherwise}. 
\end{cases}
\] (1)

The global quantity of main interest is the fraction of sites with spin 1, the «magnetization». Normally, the initial configuration is set up by randomly assigning 0 or 1 to each cell with a certain probability, in the simulations performed here it was chosen to be 0.5. The magnetization at time \( t + 1 \) is plotted against the magnetization at time \( t \), a so-called return map. The result is a noisy limit cycle, which gets less noisy as the system size grows. The limit cycle does not seem to shrink with increasing system size, and there are no experimental evidences for this to be a transient. For the limit cycle, there exists a winding number \( w = -0.3371 \), defined as the number of revolutions made per time step, the minus sign indicating that consecutive plots will describe a clockwise motion in phase space. The intrinsic noise was found to be stochastic, in the sense that the nearest-neighbour distance of points in the limit cycle goes like \( M^{-1/2} \), where \( M \) is the number of points in the plot. The system is also very stable against external noise and changes in the initial configuration.

How does the temporal behaviour of the magnetization depend on the initial configuration? Time series analysis has been used for these systems to characterize the periodicity [4], but until now no systematic investigation of the transient has been performed. Another important question is how the oscillations can be maintained over large times and in very large systems. After a short transient, the attractor is reached. If there is a synchronizing mechanism, i.e. if all cells cooperate to create the global behaviour, it should take rather long time for the oscillations to be built up, since no communication is faster than one lattice spacing per time step. One would thus expect that the exact position in the attractor that the system reaches after a fixed number of iterations would depend on the size of the system. To measure this effect quantitatively, we introduce an angle \( \dot{\phi}(t) = \arctg((m(t + 1) - 0.3)/(m(t) - 0.3)) \) and measure its distribution for different initial configurations of the same magnetization. We simulated the automaton of eq. (1) for sizes up to 512 \(^3\) on a Connection Machine CM-2. In fig. 1 the distribution of \( \dot{\phi}(500) \) for different system sizes is plotted. Note that the \( x \)-axis is very stretched. Even for the smallest systems in the figure, 32 \(^8\), the distribution is accurate. For a small system, though, the probabilities for different oscillating structures to occur are dependent on boundary effects. This is a very natural assumption which accounts for the small but noticeable aberration of the distribution for \( L = 32 \) in fig. 1. The larger the system is, the more predictable the evolution of the magnetization is. Thus, it seems that for an infinite system there is exactly the same time sequence for the magnetization for each initial configuration with given initial magnetization and correlations. The underlying mechanism seems to be local, and interactions between subsystems less important. That would also explain why the intrinsic noise is stochastic, since it is then just the result of summing the magnetization over all the subsystems. All this indicates that the quasi-periodicity is intrinsic to a small system, characterized by just a few variables, and not due to some collective synchronization of the entire system.

Which are the relevant variables of the system? We have already mentioned the magnetization. The structures of zeros and ones that can be seen in a snap-shot of the system [3] lead us to introduce a set of point correlation functions \( c_k \) by summing zeros surrounding a spin zero and ones surrounding a spin one. Let \( N \) be the number of sites in the system. For each correlation function, all the \( S(k) \) sites that are \( k \)-th nearest neighbours of
Fig. 1. – The distribution of the angle \( \phi(t) \) after 500 time steps for different lattice sizes \( L \). For \( L = 512 \) (\( \bigcirc \)), \( L = 128 \) (\( \cdots \)) and \( L = 32 \) (\( \longrightarrow \)), we made approximately 90, 1100 and 20 000 simulations, respectively. The \( x \)-axis is in radians.

the central site are taken into account:

\[
c_k = \frac{1}{NS(k)} \sum_i \sum_n S(k) \delta n_{x_i}.
\]  

(2)

These neighbouring sites form a shell surrounding the central site. For a better visualization, the correlation functions and the magnetization are plotted for every third time step in fig. 2a). It can be seen that the correlation functions have the same quasi-periodic behaviour as the magnetization, but with a phase shift of \( 2\pi/3 \). The shapes of the curves, however, are not the same. The figure also shows that the amplitude of the correlations decreases with increasing order. It therefore seems quite natural that these correlations characterize the (random) nature of the configuration of a subsystem that at the end determine the entire quasi-periodic sequence of its evolution.

It is of interest to try to describe the system with a mean-field approach, using these variables in order to calculate the global properties one measures in the real system. Several attempts have already been made [5, 6], the most successful perhaps being [7]. Here we will try a direct, pure phenomenological approach. Encouraged by the evidences of a deterministic evolution of the magnetization and the correlations, we make an ansatz that the magnetization at time \( t + 1 \) can be described as a polynomial \( P_0(m(t), c_1(t), c_1(t), \ldots) \) of all permutations of the magnetization and the correlation functions at time \( t \). Equivalently, there will be polynomials \( P_k(m(t), c_1(t), c_2(t), \ldots) \) as expressions for the correlations. These polynomials do not explicitly depend on the time \( t \), but moreover tell us how to go from \( t \) to \( t + 1 \). For each \( P_k \), we have to solve an over-determined system of equations, and we use the least-square method to find the coefficients. The result can be written as a set of iterative
functions. An example for the resulting equations for the magnetization and the first correlation functions (the coefficients have been rounded to three decimals) is

\[
\begin{align*}
    m(t+1) &= 0.718 + 0.578c_1(t) - 0.580c_2^2(t) - 3.016m(t) + 1.883m(t)c_1(t) + 2.891m^2(t), \\
    c_1(t+1) &= 0.660 - 1.660c_1(t) + 1.499c_2^2(t) - 1.421m(t) + 3.472m(t)c_1(t) - 0.188m^2(t).
\end{align*}
\]

These equations are of course of fixed-point type, and can only describe the behaviour in a region close to the attractor. The transient therefore cannot be described with this kind of equations. As a comparison to the real system, we have plotted in fig. 2b) the real limit cycle (points) together with mean-field iterations taking one (the inner line) and two (the outer line) correlation functions into account. Both mean-field iterations are quasi-periodic with a period close to three. It is striking how well the limit cycle in fig. 2b) is described by just three variables. If the values of these mean-field variables are known at some time \(t\), we can thus foresee the state of the system at time \(t+1\). This very much corroborates our picture.

![Fig. 2.](image)

**Fig. 2.** a) Plotted are the magnetization (top) and the correlations \(c_1\) to \(c_4\) (one below the other) as functions of time, showing only every third time step measured from numerical simulations of the CA with \(L = 128^3\). b) The inner solid line is the mean-field iteration including the nearest-neighbour correlation (eq. (3)), the outer line also includes next-nearest-neighbour correlation. The points come from a direct simulation of the CA for \(L = 256^3\).

Let us finally return to the original question. The reason why these systems violate the statements in [1] is rather simple: the global behaviour seen here is not a collective phenomenon. We have seen that any system larger than a given size—about 30 lattice spacings—has a well-defined trajectory in a reduced «phase space», in which each configuration is random in its detail, but characterized by a finite set of correlation functions. Each subsystem of a larger system oscillates «on its own», with fluctuations around an exact frequency as a result of small statistical fluctuations in the value of the correlations, due to their random nature. An average taken over all the subsystems will thus be more precise the larger the system is. Since there are no well-defined collective states of the system, the arguments of [1] do not apply. The system is simply composed of essentially uncoupled local oscillators, so coupled map lattices do not seem to be the adequate description either. (This point is further developed in [8], where it is shown that there are CA consisting of separated parts, each part consisting of trivial oscillating patterns. Added together they form a global behaviour.) The picture described here seems to be the key for understanding the rather
puzzling phenomenon of global oscillations found not only in high-dimensional automata, but also in many natural phenomena like the Belousov-Zhabotsinski reaction.

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