A SIMPLE MODEL WITH STRONG ASYMMETRIC COUPLINGS

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In this paper an Ising model on a random lattice with strongly asymmetric couplings inspired by neural networks is studied. We investigate the phase space structure and find evidence for an ultrametric, "multivalley" structure as observed in disordered magnetic systems. We have calculated the size of the basins of attraction.

Keywords: Spin Glass; Asymmetric Models; Neural Networks; Ising Models.

1. Introduction

Based on the study of disordered magnetic systems, various types of spin glasses were proposed to realize the basic functions of neural systems (in particular the human brain): the memory or the capacity to recall a lived experience; and learning, or the capacity to associate these lived experiences.

Usually Hopfield’s model is associated with the version in which the connection between sites are symmetric. If we call \( J_{ij} \) the connection between neurons \( i \) and \( j \), the symmetric version implies \( J_{ij} = J_{ji} \). The assumption of symmetric connections has been criticized from the biological point of view (if the neuron \( i \) influences the neuron \( j \) there is no reason why the neuron \( j \) should influence the neuron \( i \)). In this paper we study the effects of random asymmetry on a very simple model.

In the Hopfield paper some arguments were presented to show that weak asymmetry does not qualitatively change the "retrieval capacity" of the system. This result was later corroborated by Amit et al. and Sompolinsky and Kanter. On the other hand, Parisi has conjectured that random asymmetry of the synaptic matrix could eliminate the spin glass phase. Crisanti and Sompolinsky have studied in

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detail asymmetric spin glasses and have corroborated Parisi's conjecture. Only in
the Ising model with total asymmetry at $T = 0$ should metastable states remain.
Spitzner and Kinzel\cite{7} have studied the Hopfield model for directed connections and
calculate the critical storage capacity $\alpha = 0.078 \pm 0.001$. The main effect of di-
rectional connections is that of diluting the bonds. In the Hopfield model with
asymmetric connections they have observed chaotic states in the dynamics (even at
zero temperature). Rieger and Schreckenberg\cite{8} have studied a one-dimensional spin
glass with antisymmetric connections ($J_{ij} = J$ or $-J$ with probability $p$ or $(1 - p)$).
The dynamics of the chain is defined by the existence of ferromagnetic segments
(1D "clusters") which are coupled by antisymmetric bonds.

In this paper we propose the simplest two-dimensional model in order to study
the effect of asymmetric couplings. Consider $N$ sites on a square lattice of length
$L$ ($N = L \times L$). Each site is occupied by Ising spins, with two states. We define
strong asymmetric couplings (directed couplings) $\{J_{ij}\}$ randomly distributed to be
either $+1$ or $0$ with the condition

$$\sum_i J_{ij} = 2,$$  \hfill (1.1)

which means that each site $i$ is connected to and influences exactly two of its four
neighbours. We assume that $J_{ij}$ and $J_{ji}$ are independent variables, that is, if spin
$i$ influences spin $j$, this does not imply that spin $j$ influences spin $i$. The circuits
built by these directed couplings on the lattice may be closed or open, depending
on the coupling configuration $\{J_{ij}\}$. Figure 1 shows a possible configuration with
these features (we assume periodic boundary conditions). No noise is considered
in the dynamics (we have a purely deterministic evolution), which is analogue to a
$T = 0$ calculation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{network.png}
\caption{A sample of the network of unidirectional coupling connections ($L = 4$) and periodic
boundary conditions. The arrow indicates the connection between spins ($J_{ij} = 1$ or 0).}
\end{figure}

We now show the dynamics of the model, present our results and then conclude.
2. Dynamics of the Model and Results

Given an initial state of the spins $C_I\{S_i\}, i = 1, \ldots, N$ the system evolves to final state $C_F\{S_i\}$ by the dynamics:

$$S_i(t + 1) = \text{sign}(h_i(t)), \quad (2.1)$$

where $S_i(t)$ is the state of spin $i$ at time $t$ and $h_i(t)$, the "action" of neighbouring spins $j$ on spin $i$ given by:

$$h_i(t) = \sum_j J_{ij} S_j(t). \quad (2.2)$$

When $h_i(t) = 0$, we choose that $S_i(t + 1) = S_i(t)$. The iteration stops when $C\{S_i(t + 1)\} = C\{S_i(t)\}$.

We compared two processes: sequential and random updating. In the first case we update the $N$ spins line by line, in the second case $N$ spins are chosen randomly at each updating step. Parallel updating was not considered. In simulations of large $N$ spin glasses only for sequential updating a remanent magnetisation was obtained. In our study we observe that in both cases, random and sequential updating, in general the model converges to stable final states $C_F\{S_i\}$. However, as previously pointed out, there could be configurations of couplings where the system does not converge to a final stable state. In this case, a cycle of states is expected as the attractor of the dynamics. However, the set of basins of attraction is smaller in the case of random updating. Sequential updating results in a symmetric distribution of the basins of attraction, as shown in Fig. 2. The final state number (horizontal axis) is the value of the $N$ bits word in which each bit corresponds to a spin on the lattice ($\text{bit}_i = 1$ for $S_i = 1$ or $\text{bit}_i = 0$ for $S_i = -1$). The $-1$ ferromagnetic state has state number $0$, for instance. We observe that the weights of final states $C_F\{S_i\}$ and $C_F\{-S_i\}$ are equal. Weight means the percentage of initial states of spins evolves to a specific final state. Despite the differences between the weights of final states, the same final states of spins obtained in the random updating are also obtained in the sequential updating for same coupling configurations. In the following we chose to use sequential updating.

Networks with 9, 16, 25, 36 and 49 nodes were considered ($L = 3, 4, 5, 6$ and $7$). Small sizes were chosen, to allow for the study of the trajectories in phase space of all possible initial states $C_I\{S_i\}$, for $L = 3, 4$, on a set of different coupling configurations $\{J_{ij}\}$ and for a high number of initial states for $L = 5, 6$ and $7$. For $N$ sites we have $2^N$ possible initial states of spins and $6^N$ possible configurations for the distributions of $\{J_{ij}\}$. The number of coupling configurations considered for each size of the network ranged between $10^4$ and $10^5$. About 190 hours of CPU time were spent on a C220 Convex computer in this calculation.

One important feature of systems with spin glass phase is the ultrametricity in the phase space of the final states. In order to study the phase space structure of the final states of this model we calculate the Hamming distance between final states $C_F\{S_i\}$ for a set of coupling configurations $\{J_{ij}\}$. Following Sourlas, we calculate
Fig. 2. Distribution of weights (sizes) of the various attractor basins of a simulation on one network of 9 sites ($L = 3$) for a specific coupling configuration. We have started with all of the 512 possible initial states of spins. Only 15 of these were reached as final states for a specific coupling configuration.

the density distribution of triangles $P((d_{\text{max}} - d_{\text{mid}}), d_{\text{min}})$, where $d_{\text{max}} \geq d_{\text{mid}} \geq d_{\text{min}}$ are Hamming distances between any set of three final states. Our results are shown in Fig. 3. We observe that this density grows along the line $(d_{\text{max}} - d_{\text{mid}}) = 0$ when the systems grow. As pointed out by Sourlas, this is a strong evidence for an ultrametric structure of the phase space.

We have calculated the distribution of the weights $\omega_k$ of the final states $C_F^k\{S_i\}$ for a given configuration of couplings $\{J_{ij}\}$, defined as:

$$\omega_k = \frac{n(C_F^k\{S_i\})}{n_{\text{total}}}, \quad (2.3)$$

where $n(C_F^k\{S_i\})$ is the number of initial states $C_I\{S_i\}$ which have converged to the particular final state $k$ and $n_{\text{total}}$ is the total number of initial states considered in the simulation. From the definition it follows

$$\sum_k \omega_k = 1.$$

We calculate the function $Y$ defined as:

$$Y(\{J_{ij}\}) = \sum_k \omega_k^2 \quad (2.4)$$

for different configurations of $\{J_{ij}\}$ and the average $\bar{Y}$ over the different $\{J_{ij}\}$. This function describes the probability that two randomly chosen initial states $C_{I,1}\{S_i\}$ and $C_{I,2}\{S_i\}$ converge to the same final state of spins. As pointed out by Derrida, $Y$ tells us whether there are big or small valleys. If $Y = 1$, this means that there is only one big valley with weight $\omega_1 = 1$. If $Y \approx 0$, there are many valleys each of which has a small weight. If $Y$ decreases one has fewer big valleys and more small ones. Since $Y$ depends on the $\{J_{ij}\}$ the meaningful function to consider is the
A Simple Model with Strong Asymmetric Couplings

Fig. 3. Density of triangles $P((d_{\text{max}} - d_{\text{mid}}), d_{\text{min}})$, for various lattice sizes. (a) shows the calculations for $L = 3$, (b) for $L = 4$ and (c) for $L = 5$. For $L = 5$ we observe the largest density along the $(d_{\text{max}} - d_{\text{mid}}) = 0$ line.

probability distribution $\Pi(Y)$. In Figs. 4a, 4b, 5a, 5b and 5c we show the probability distribution $\Pi(Y)$ for networks of sizes $L = 3$, 4, 5, 6 and 7. For $L = 3$ and 4 we calculate $\Pi(Y)$ for two different sets of initial states. The histogram drawn in white bars represents the results obtained when we make the simulation with all possible initial states of spins for a set of different configuration couplings $\{J_{ij}\}$ (for $L = 3$, $n_{\text{total}} = 512$ and for $L = 4$, $n_{\text{total}} = 65,536$ for each $\{J_{ij}\}$). For the histograms in hatched bars in Figs. 4a and 4b (as well in Figs. 5a, 5b and 5c) we choose a fixed number of random initial states of spins. We observe that the figures have the same shape and that $\bar{Y}$ decreases for increasing system size. In Table 1 we show the values of $\bar{Y}$ for different network sizes. Figure 6 shows the logarithm of $\bar{Y}$ against the size $N$ of the lattice. We observe that the value of $\bar{Y}$ decreases exponentially with $N$:

$$\bar{Y} \sim e^{-\alpha N},$$

with $\alpha \simeq 0.11 \pm 0.03$. 
Fig. 4. Probability distribution $\Pi(Y)$ (see text). White bars represent the $\Pi(Y)$ distribution for all possible initial states for $L = 3$ (512) from 100,000 random coupling configurations and for $L = 4$ (65,536) (4b) from 1,000 random coupling configurations. Hatched bars represent the histogram obtained when we have started with 500 (for $L = 3$) and 10,000 (for $L = 4$) random initial states for the same set of random coupling configurations.

Fig. 5. Probability distribution $\Pi(Y)$ for (a) $L = 5$ from 10,000 random initial states for 1,000 random coupling configurations; (b) $L = 6$ and (c) $L = 7$ from 50,000 random initial states for 300 random coupling configurations.
Fig. 6. Plot of $\ln(\bar{Y})$ against the size $N$ of the lattice. The dotted line represents a linear regression obtained from the averaged values (circles).

### Table 1. Values of $\bar{Y}$ for five sizes $L$ of the network.

<table>
<thead>
<tr>
<th>Size</th>
<th>$\bar{Y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.189 ± 0.101</td>
</tr>
<tr>
<td>4</td>
<td>0.0596 ± 0.0511</td>
</tr>
<tr>
<td>5</td>
<td>0.0107 ± 0.0160</td>
</tr>
<tr>
<td>6</td>
<td>0.0025 ± 0.0039</td>
</tr>
<tr>
<td>7</td>
<td>0.0037 ± 0.0068</td>
</tr>
</tbody>
</table>

From the previous discussion we conclude that our model has a multivalley structure, with many valleys, all of small weight. The same behaviour was observed in the spin glass phase.

In Fig. 2 we can observe that the "ferromagnetic" states (in which all spins are equal) have a greater weight than other states. We have calculated the weight of these states ($\omega_{\pm1}$) for different sizes and found an exponential decay

$$\omega_{\pm1} \sim e^{-\alpha_{\pm} N}, \quad (2.6)$$

with $\alpha_{\pm} \approx 0.08 \pm 0.01$. This result is presented in Table 2. The values of $\omega_{\pm1}$ are averages over many configurations of couplings.

In both tables and in Figs. 6 and 7 we observe a different behaviour for the data for the largest system ($L = 7$). This could be caused by our choice of small set of initial states. We have observed fluctuations in mean values when we choose different sets.

In order to understand this behaviour in a model with diluted "ferromagnetic" couplings, we have calculated the mean fraction of "frustrated" spins for different
Table 2. "Ferromagnetic" weights for varying sizes $L$ of the network.

<table>
<thead>
<tr>
<th>Size</th>
<th>$+1 &amp; -1$ weights ($\omega_{\pm1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.529</td>
</tr>
<tr>
<td>4</td>
<td>0.262</td>
</tr>
<tr>
<td>5</td>
<td>0.0897</td>
</tr>
<tr>
<td>6</td>
<td>0.0366</td>
</tr>
<tr>
<td>7</td>
<td>0.0429</td>
</tr>
</tbody>
</table>

Fig. 7. Fraction of "frustrated" spins (7a) and "independent" sites (7b) against the inverse of the size of the lattice. The dotted line represents a linear regression from the averaged values (circles).

sizes of networks. Let us define the function $g_i$

$$g_i = \sum_j |J_{ij}S_j|.$$  \hfill (2.7)

We call a spin $i$ "frustrated" if the local field $h_i = 0$ and $g_i \neq 0$. This means that its spin $S_i$ cannot satisfy the action of its neighbours. In Fig. 7a we show the average fraction of "frustrated" spins ($<S_f>$) for the stable final states obtained in previous calculations. We observe in this figure a tendency towards a constant fraction of "frustrated" spins (for $N = 25$ and $36$ the values of $<S_f>$ are nearly the same). Otherwise, we call a site "independent" if $h_i = 0$ and $g_i = 0$. These "independent" sites receive no influence at all from their neighbours. The existence of these "independent" sites does not depend upon the particular states $C_T\{S_i\}$ of the spins, but only upon the distribution of the couplings $\{J_{ij}\}$. They are not changed by the dynamics. We have calculated the fraction of "independent" sites $<S_{\text{ind}}>$ for a sample of stable final states and the results are shown in Fig. 7b. As observed in Fig. 7a, we also have observed here a tendency to constant fraction of "independent" sites. From this, one has two ways to create different attractor basins in the system: the "independent" sites and "frustrated" spins are the features that allows the existence of different signal ferromagnetic clusters.
3. Conclusions

In this paper we have studied a directional model with a strong asymmetry in the coupling connections without frustration in the usual sense (negative $J_{ij}$). This model was built inspired on the discussion of the basic features of neural networks. We obtain numerical evidence that this model has a multivalley structure. The probability distribution of triangles in the phase space of final configurations shows indications for ultrametricity. The calculation of $\Pi(Y)$ shows a phase space with a large number of final configurations of similar weights.

The lattice can have open or closed circuits. The calculation of "frustrated" spins and "independent" sites shows that there are "disjoint ferromagnetic clusters" (closed circuits) and "directional clusters", which do not influence each other. However different (up or down) "directional clusters" can be connected by a same site ("frustrated spin") and the spin on this site cannot satisfy the simultaneously influences of these clusters. This is the picture that allows us to understand the existence of different final configurations in a model having only ferromagnetic couplings.

From the stabilization rule $S_i(t+1) = S_i(t)$ when $h_i(t) = 0$ one does not have any fluctuation in the evolution of the model. With the feature of existence of "independent" sites and "frustrated" spins it could grow up the degeneracy of the model and the system could not in a local minimum. It would be important to introduce noise into the model (as the temperature in ergodic thermodynamic systems, for instance) and study the behaviour of the basins of attraction. In some disordered models a small noise can destroy the multivalley structure of phase space. We have observed that for any coupling matrix the initial spin configuration does not always evolve to a stable final one. The existence of periodic attractors or a chaotic regime is an important feature to be investigated in our model.

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References