Patterns and Scaling in Fracture

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Abstract
The fracture of solids is regarded through a statistical physicists eye. Considering rather basic models we can give evidence for the appearance of fractal crack patterns. The breaking characteristics, i.e. the relation between force and displacement during rupture, can be obtained for these models. As function of the system size the breaking characteristics follows a scaling law. Just before the system fully breaks apart the distribution of local strain is multifractal.

1 Introduction
The fracture of a solid under an external load is a problem of much technological importance and has been intensively studied for the last hundred years [1]. On very small length scales (≤ 10⁻⁶ cm) a fracture is a subject of material science. From the electronic level [2] to the level of dislocations or grain boundaries [3] the mechanisms of fracture are highly material dependent. On very big length scales (> 10⁻¹ m) the prevention of fracture is studied by engineers. Their results are mainly based on experience and depend essentially on the application and the shape of the sample. On intermediate length scales the behavior of the solid can be described by the methods of applied mechanics, i.e. by continuous equations of motion. There exist on this level just few types of different behaviors - like elastic, plastic, viscoelastic. If the reader wants to know more about the various cases I recommend Ref. [4].

Here we will restrict our attention to elastic media, i.e. brittle fracture, described by equations of motion. We will show some novel concepts that statistical physics has contributed to the description of fracture. On one hand the introduction of disorder by considering material constants to be spatially random functions has been very useful. On the other hand, meanwhile standard notions in critical phenomena, like scaling, fractals or multifractals have helped to formulate new, rather universal laws. After a brief survey of the basic notions of elasticity in the next section, which should be skipped by the experts, we will consider the growth of one single crack. In analogy to fluid fingering we formulate crack growth as a moving boundary problem in Section 3. In Section 4 we will present the various ways how crack growth can be modelled on a lattice. We show how one obtains deterministic fractal cracks on a lattice in Section 5. In Section 6 we introduce quenched disorder and show how the interaction of cracks results in the scaling of the breaking characteristics and multifractality in the local distribution of strain. We conclude in Section 7.

2. Some basic notions of elasticity
In order to describe fracture one needs to know first how an unfractured solid responds to an externally applied force. For this purpose let us consider the Young experiment, i.e. a homogeneous block of size l × w × l submitted to a uniaxial force F. For small forces one expects a linear response of the type [5]

\[ \sigma = \frac{F}{w^2} = E \frac{\Delta l}{l} = - \frac{E}{v} \frac{\Delta w}{w} \]  

(1)

where E is called the Young modulus, v the Poisson ratio and \( \sigma \) the stress.

The formation and growth of a crack crucially depends on the local stress field. It is therefore important for the following to know the equations of motion of an elastic medium. A good variable to describe the medium is the displacement field \( u \) which gives for each volume element its position with respect to a reference position - usually the equilibrium position it has without externally applied forces. Since one wants to have a theory that is translationally invariant only the derivatives of \( u \), i.e. the tensor \( \partial_i u_j \), is going to appear. This tensor can be split into a symmetric part, called the strain \( \varepsilon_{ij} \), and an antisymmetric part. The antisymmetric part describes local rotations of a volume element. In a homogeneous medium a single volume element cannot be rotated with respect to the rest, so that in this case only the symmetric part will be relevant. Porous or damaged media are not homogeneous and it is therefore also useful to consider the non-symmetric theory, also known as Cosserat theory.

Since we want to describe linear elasticity the strain is given by

\[ \varepsilon_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \]  

(2)

Let us define \( \mu \) and \( \lambda \) which are positive dimensionless material constants called Lamé coefficients that are in fact related to the Young modulus \( E \) and the Poisson ratio of eq. (1) through

\[ \lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)} \]  

(3)

and since \( \lambda \) and \( \mu \) are positive one finds \(-1 < \nu \leq \frac{1}{2}\). The conjugate variable to \( \varepsilon_{ij} \) is the stress tensor \( \sigma_{ij} \) for which one obtains

\[ \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \delta_{ij} \sum \varepsilon_{\gamma\gamma} \]  

(4)

This linear relation is called Hooke’s laws. One obtains the equation of motion for the displacement field

\[ \nabla (\nabla \cdot u) + (1 - 2\nu) \Delta u = 0 \]  

(5)

which is called Lamé or Navier equation. This equation is just a consequence of the invariance under translations and coordinate transformations, imposing linearity and the homogeneity and isotropy of the medium.

In order to allow for local rotations in the medium one
considers in asymmetric (or Cosserat) elasticity [6] that each volume element has an additional angular variable $\varphi$. The strain is then defined as $\varepsilon_{sl} = \partial_i u_g - \sum_s e_{sp} \varphi$, where $e_{sp}$ is the totally antisymmetric tensor and one has in addition a torsion tensor $\kappa_{sl} = \partial_i \varphi_l$. Analogously to the symmetric elasticity conjugated variables stress $\sigma_{sl}$ and moment $\mu_{sl} = \partial E/\partial \kappa_{sl}$ can be defined. They fulfill linear relations

$$\begin{align*}
\sigma_{sl} &= (\mu + \lambda)\varepsilon_{sl} + (\mu - \lambda)\varepsilon_{kl} + \lambda \delta_{sl} \sum \varepsilon_{tn} \\
\mu_{sl} &= (\gamma + \delta)\kappa_{sl} + (\gamma - \delta)\varepsilon_{kl} + \eta \delta_{sl} \sum \kappa_{tn}
\end{align*}$$

(6)

with six material constants of which $\alpha$ and $\delta$ describe the asymmetry. The equations of motion in equilibrium without internal body forces, called Cosserat equations, are then given by

$$\begin{align*}
(\mu + 2\lambda)\Delta u + (\lambda + \mu - \alpha)\nabla(\nabla u) + 2\alpha \nabla \times \varphi &= 0 \\
(\gamma + \delta)\Delta \varphi + (\beta + \gamma - \delta)\nabla(\nabla \varphi) + 2\alpha \nabla \times u - 4\alpha \varphi &= 0.
\end{align*}$$

(7)

We see that for $\alpha = 0$ one recovers the Lamé equation. In two dimensions one has $u = (u_1, u_2, 0)$ and $\varphi = (0, 0, \varphi)$ so that one has three independent variables per site determined by three equations of motion.

3. Fracture as a growth model

In order to study how a void in the material becomes a crack and how the crack grows it is useful to formulate fracture as a moving boundary problem. For simplicity we will do this here only for the case of symmetric elasticity.

Let us consider an elastic medium $B$ subjected at the outer boundary $A$ to some externally imposed displacement $u_0$ (see Fig. 1). Inside the medium one has a void (or crack) $D$ so that on its surface $C$ the elastic medium feels no restoring forces in direction of the void. Consequently one has the boundary condition on the surface of the crack that the stress $\sigma_1$ perpendicular to the crack vanishes: $\sigma_1 = 0$. Due to eqs. (3) and (6) this is a condition on the derivatives of the displacement field. Since the Lamé equation is elliptic the two boundary conditions uniquely define one equilibrium solution for the displacement field.

The dynamics of crack growth can be described by a velocity $v_a$ of the surface of the crack in perpendicular direction. This velocity $v_a$ depends on the microscopic mechanisms of the rupture of the material and no explicit expression for it is known. It is in particular probably not possible to derive such an expression from first principles or to find a material-independent form for it. Within our framework of symmetric elasticity the only non-zero first derivative of $u$ on the boundary of the crack is $\sigma_1$, the stress parallel to the surface of the crack. Therefore to this order $v_a$ is a function $F$ of $\sigma_1 - \sigma_2$ where $\sigma_2$ is the cohesion force which essentially means that one considers mainly rupture through cleavage. In order to take into account the very common rupture due to bending, that occurs for instance under the action of a cantilever, one would have to take into account either higher derivatives of $u$ or, better, asymmetric elasticity. If one wants to make a concrete calculation one has to make assumptions on the explicit form of the function $F$. In the following we shall use $v_a = c(\sigma_1 - \sigma_2)$ where $c$ and $\sigma_1$ are material constants and we will often set $\eta = 1$.

Let us consider the two dimensional case. Then the moving boundary problem can be formulated as following: the medium follows the equations

$$V(\nabla \cdot u) + (1 - 2\nu)\Delta u = 0$$

(8a)

with the boundary conditions

$$u_a = u_0$$

(8b)

on the external boundary and

$$\partial_1 u_1 + \partial_1 u_2 = 0$$

$$1 - 2\nu \partial_1 u_1 + \nu \partial_1 u_2 = 0$$

(8c)

(8d)

on the internal boundary. The latter boundary, which is the crack surface, moves with a velocity $v_a = c(v_1 \partial_1 u_1 + (1 - \nu) \partial_1 u_2 - \sigma_1)$ normal to the surface. In eq. (8c) which is identical to $\theta_1 = 0$ and eq. (8d) we replaced stresses by displacements using eqs. (2) and (4). The symbols $\perp$ and $\parallel$ always denote the components perpendicular and parallel to the surface of the crack. Moving boundary problems similar to the one of eq. (8) appear in various contexts of physics like dendritic growth or viscous fingering and are very difficult to treat mathematically. This comes from the fact that after moving the boundary the internal boundary condition changes, changing the solution and therefore also the speed of the boundary. In other words a given solution is only valid during an infinitesimally short time and one must iteratively solve the equations and then move the boundary in principle an infinity of times to see the precise development of the crack.

Essentially two techniques have been applied to moving boundary problems: stability analysis and numerical solution in discrete time steps. The stability analysis which we discuss briefly in the following only considers one iteration, i.e. what will happen after an infinitesimally short time. The numerical solution can be performed in various ways, as will be discussed in the next section.

Suppose one knows one particular, simple solution of the moving boundary problem in which the shape of the boundary does not change in time (stationary solution) so that the problem is similar over all times. The linear stability analysis consists in investigating the effect of a small perturbation on this stationary solution in lowest order. If the perturbation
Fig. 2. Patterns produced by water pushed into a bentonite/water paste at concentrations: 0.08 at left and 0.20 at right; taken from Ref. [8].

increases in time one has an instability which will give rise to more complicated but also more interesting solutions. Although these new solutions cannot be obtained explicitly one knows at least the motor driving its evolution, namely the instability. Some of these instabilities are very well-known like the Saffman-Taylor instability in viscous fingering, the Mullins-Sekerka instability in dendritic growth or recently the tip-splitting instability [7] which drives the growth of diffusion limited aggregates.

One type of stability analysis for fracture can be performed when the cohesive force is just compensated by the externally applied load. In this case all modes $k$ of a perturbation of the form $\exp(ik\rho + \omega t)$ on the radius of a circular hole will be unstable, i.e. their $\omega$ is positive and the perturbation increases with time.

This is to be compared to the situation encountered in the analogous scalar moving boundary condition problem for which the medium is described by the Laplace equation. A scalar medium can describe an electrical field in dielectric breakdown, pressure in viscous fingering, temperature in solidification, an electrochemical potential in electrodeposition, particle density in diffusion limited aggregation, and so on. For viscous fingering the underlying moving boundary value problem is only unstable for perturbations with wavevectors $k$ exceeding a certain critical value that depends on the surface tension [7].

The difference between viscous fingering and fracture resulting from the above observation can be seen from the beautiful experiments performed by Van Damme and collaborators [8]. They pushed water into clay, which is a very viscous, Non-Newtonian fluid and on short time scales behaves like a solid. A simple model for such a behaviour could be a "viscoelastic fluid" [9]. In Fig. 2 we show the patterns formed by the penetrating water. The experimental set-up is the same, only the concentration is different between the two sides of the figure. On the left we have viscous fingers with a finite radius of curvature at the tips and on the right we see fracturing. The crack tips are sharp because, as discussed above even perturbations of very small wavevector $k$ are unstable. The left pattern has a fractal dimension of about 1.6, while it is about 1.4 for the pattern of the right. If the density is slowly increased a morphological transition between the two patterns can be witnessed.

4. Modelisation of fracture on a lattice

A numerically tractable, formulation of fracture, is given by discretizing the continuum equations. In this case the medium is reduced to a set of points embedded into a grid. Only local laws, like the balance of forces and momenta, are considered and their implementation involves for each point only a few neighbors. Mathematically the calculation of collective properties like the equilibrium displacement field is then reduced to solving a set of coupled linear equations [10].

The simulation of a rupture process must be done in an iterative way: the equations must be solved in order to determine which bond should be broken, but once the bond has been broken the (internal) boundary condition and therefore also the solution of the equations is changed. Consequently the equations must be solved again if one wants to know which bond to break next and so on. We see that the process is rather heavy because the set of linear equations must be solved as many times as bonds are broken. With the present computational means one can break one or two thousand bonds in a lattice of roughly $10^6$ sites.

The algorithm performed for each iteration can be decomposed into five steps:

1. the set of equations is solved,
2. the set of all the bonds that are eligible to be broken is determined,
3. for each bond of this set a certain quantity $p$ is calculated using the solution of the equation on the sites adjacent to the bond,
4. according to a rule that depends on $p$ one chooses, out of the set, one bond to be "broken",
5. the bond is broken, i.e. its elastic modulus is changed in the equations.

Each of these steps allows for a large variety of options that can take into account many possible physical situations.

1. describes the nature of the medium and the externally applied constraints, (2) the connectivity of the crack, (3) and (4) the breaking rule and the disorder and (5) allows to incorporate a residual strength. In the following we will discuss each of the steps in more detail.

The problem of the breakdown of a network of electrical fuses is very similar to fracture only that the variable one has instead of the vectorial displacement field $u$ is the scalar electrical potential $\phi$. Since the scalar nature of the problem simplifies numerical and analytical work this analogon is often studied too. In fact, in two dimensions, the "dual" [4] of the fuse problem is dielectric breakdown [12]. In scalar models one studies the Laplace equation $\Delta \phi = 0$.

For vectorial models rather different discretizations for the equations have been in use: the central force model [14], the bond-bending model [15] and the beam model [16]. The beam model is a rather straightforward discretization of the Cosserat equations. In two dimensions there are, following the remark after eq. (7), on each site $i$ three continuous degrees of freedom: the two coordinates $x_i$ and $y_i$ and $z_i = l - \varphi_i$, where $\varphi \in [-\pi, \pi]$ is the rotation angle and the lattice spacing $l$ is usually set to unity. Nearest neighboring sites are connected through a "beam" in such a way that it joins site $i$ forming an angle $\varphi_i$ with respect to the underlying undistorted lattice. In Fig. 3a we see the form of the beams when just the site in the middle is rotated. Each beam is assumed to have a finite cross section $A$ so that besides a traction force $f$ along its axis one also has a shear $s$ and moments $m_i$ and $m_j$ at its two ends $i$ and $j$. These can be calculated [17] in linear (elastic) approximation. For a horizontal beam one obtains in two
slope in Fig. 4, the presence of disorder can now easily be implemented by allowing the bonds of the same lattice to have different constitutive laws. Strength fluctuations can be represented by choosing the threshold value for each bond randomly [22]. Spatial variations in the elastic moduli can be described by randomly choosing for each bond different slopes in Fig. 4 [11].

The most prominent case of spatial density fluctuations are porous media where a given point in space is either massive or empty. This situation can be modelled by randomly taking out a bond of the lattice with a probability \( q = 1 - p \) before the breaking process starts and has been extensively studied for electrical [13, 23] and mechanical [24] networks. Particularly interesting is the behaviour of these dilute systems close to the percolation concentration \( p_c \).

While in the case of dilution the probability distribution is binary the strength and conductivity fluctuations are generally chosen to be continuously distributed. Most often a uniform distribution,

\[
P(x) = 1 \quad \text{with} \quad 0 \leq x \leq 1
\]

is chosen in this case, but also powerlaw,

\[
P(x) = (1 - x)^\alpha \quad \text{with} \quad 0 \leq x \leq 1 \quad \text{and} \quad \alpha > -1
\]

and Weibull distributions,

\[
P(x) = mx^{m-1}e^{-x^m} \quad \text{with} \quad 0 \leq x \leq 1
\]

have been considered [25]. The constants \( \alpha \) and \( m \) are parameters of the distribution which determine how strong the disorder is. Distributions with a lower cut off \( x_1 \), i.e. for which \( 0 < x_1 < x \), are also of interest [22]. It is not easy to distinguish experimentally between the different types of distributions. If one assumes a Weibull distribution values \( 2 \leq m \leq 10 \) have been fitted to data measured on rocks.

In quenched systems the random variables are fixed at the beginning and the following rupture process is completely deterministic. If the dominant disorder stems from thermal fluctuations it seems reasonable to use a Boltzmann factor and one approach [20, 26] is to calculate before each breaking for each bond \( i \) its elastic energy \( \varepsilon_i \) and from there \( p_i = e^{-\beta \varepsilon_i} \), where \( \beta = 1/k_B T \). The bond to be broken, \( i_b \) is then chosen with a probability proportional to its value \( p_i \).

Instead of the energy \( \varepsilon_i \), other quantities have been considered [27].

Since thermal fluctuations are not the only source for annealed disorder and since a first principles derivation of the probabilistic breaking rule is not possible at present, another approach, inspired from dielectric breakdown models [12], has been taken: The bond to be broken is chosen with a probability \( p_i \) proportional to its elongation \( \delta \) [21, 28]. This can be generalized to \( p_i \propto \delta^\eta \). The exponent \( \eta \) is phenomenological and can be adjusted to fit experimental results. In general \( \eta \) is a "relevant" parameter, i.e. the fractal dimensions depend continuously on \( \eta \).

As opposed to simple springs the elastic beams can break in various modes: A rubber band will tear apart when stretched but a glass rod will crack when bent. Try, on the contrary, to break a rubber band through bending or a glass rod through stretching! In three dimensions a third mode of rupture appears, namely torsion.

Mechanical breaking rules should reflect these various rupture modes. This can be achieved by defining the quantity
p that determines the breaking as a sum of terms each corresponding to one mode. In the quenched case for instance one can consider that the beam breaks for which

$$ p = \left( \frac{t}{t_f} \right)^2 + \frac{\max (|m_1|, |m_2|)}{t_m} $$

is largest [29]. In this criterion, which is inspired from the “von Mises yielding criterion” [5] $f$ is the force stretching the bond, $m_1$ and $m_2$ are the moments acting at the two ends of the bond and $t_f$ and $t_m$ are two material-dependent threshold values. The first term in eq. (13) describes breaking due to stretching and the second term takes into account the bending mode. If one has quenched disorder in the strength of the bonds the threshold values $t_f$ and $t_m$ are chosen randomly and one can in fact have for $t_f$ a distribution between 0 and 1 while $t_m$ may be distributed between 0 and $r$. In the case of eq. (11) one would then have

$$ P(t_f) = (1 - x)t_f^\alpha \text{ with } 0 < t_f \leq 1 $$

and

$$ P(t_m) = (1 - x)^{r-1}t_m^\alpha \text{ with } 0 < t_m \leq r. $$

Here $r$ measures the susceptibility of the material to break through bending as compared to stretching. In the case of annealed disorder one can use

$$ p = |f|^2 + r \cdot \max (|m_1|, |m_2|)$$

as the quantity to which the breaking probability is proportional.

Damage can be modelled within the breaking rule by lowering the breaking threshold of bonds in the vicinity of the crack by an amount proportional to the strain without breaking them. Bonds damaged in this way are more likely to break at one of the next iteration steps. Short-lived damage can be described by considering at iteration step $t$ the quantity [30]

$$ p' = p(t) + f_0 \cdot p(t - 1) $$

where $p(t)$ is the usual quantity one would have used at step $t$ without damage and $f_0$ is a parameter. For $p(t)$ one could for instance take expression (13) or (15). The second term in eq. (16) stimulates the memory effect due to damaging that occurred at the previous iteration step and $f_0$ controls the strength of this memory. The quantity $p'$ can be used as before either for quenched disorder by breaking the bond for which $p'$ is largest or for annealed disorder by breaking a bond with probability proportional to $p'$.

Remanent damage occurring for instance in stress corrosion is an accumulation of the damaging that occurred during the entire breaking process. It can be modelled by putting counters $c(t)$ on the bonds susceptible for damaging, like for instance the bonds on the surface of the crack. At the beginning all counters are set to zero. At each iteration step $t$, after having calculated $p(t)$, one defines $c(t) = (1 - c(t - 1))/p(t)$ and chooses the bond which has the smallest $c(t)$, namely $c_{\min}$. In the quenched case this is the bond one will break. Then all the counters are updated through [30]

$$ c(t) = c_{\min} \cdot p(t) + f \cdot c(t - 1). $$

We see that $c(t)$ sums up the damages from all previous iteration steps where the memory factor $f$ controls how strongly previous damage remains. If $f = 1$ the damage is irreversible. The case $f \to 0$ corresponds to the criterion of eq. (16) with $f_0 = 1$. More about these models will be said in the next section.

5. Deterministic growth of a fractal crack

Let us next describe a specific example for the growth of one connected crack without including yet any disorder. We consider a finite square lattice of linear size $l$, with periodic boundary conditions in the horizontal direction. On top and on bottom an external shear is imposed. We remove one beam in the center of the lattice which represents the initial microcrack. Next we consider the six nearest-neighbor beams of this broken beam. These include the two beams that are parallel to the broken beam and the four perpendicular beams that touch a common site with the broken beam. This choice of nearest-neighbors comes from the fact that the actual crack consists of the bonds that are dual to the set of broken beams. The beam model is solved by a conjugate gradient method [10] to very high precision ($10^{-20}$) and the $p$'s of eq. (15) are calculated for each of the nearest-neighbor beams. We set $p = 0$ for a beam that is not a nearest-neighbor to the crack. Now various criteria for breaking are possible: I. One breaks the beam with the largest value of $p$; II. One breaks the beam for with $p' = p(t) + f_0 \cdot p(t - 1)$ is largest (see eq. (16)); III. On each beam of the lattice we put a counter $c(t)$ which is set to zero in the very beginning and updated according to eq. (17). Then one breaks the beam for which $z(t) = (1 - c(t - 1))/p(t)$ is smallest. All three breaking criteria described above are deterministic. In criterion II the limit $f_0 \to 0$ gives criterion I.

Let us next discuss the results that one finds [30] for the above models. If one breaks according to criterion I, as shown in Fig. 5 for $L = 50$ and $r = 0.28$, cleavage tends to have the crack grow in the diagonal direction while the bending mode favours a horizontal rupture. The competition between these two effects can lead to complex branched structures. The exact shape of these cracks strongly depends on $r$ and the system size. For any finite $r$ the horizontal rupture will eventually win if the system is large enough while for $r = 0$ one obtains diagonal cracks with eventual kinks. For this reason the cracks will not be fractal. In the analogous scalar model (i.e. dielectric breakdown), however, only straight lines will be formed in criterion I; the different
behaviour here is due to the fact that competing directions are possible in a vectorial model.

Let us now consider cracks grown using criterion II. We see in Fig. 6 a crack with \( r = 0 \) and \( \eta = 1.0 \) obtained in a system of size \( L = 118 \). Over four hours on one Cray XMP processor were needed to generate one of these structures. In fig. 6 we only show the upper part of the crack, the lower part being reflection symmetric. The very slight curvature of the crack is a finite size effect of the lattice which is, however, such a weak effect that its influence cannot be noticed quantitatively for instance in the value of the fractal dimension that we will discuss next.

If we count the number of broken beams inside a box of length \( l \) around the first broken beam and plot it as a function of \( l \) in a log-log plot ("sand box method") we find straight lines with slopes larger than unity which means that the cracks are fractals. In system sizes of \( L = 118 \) we find for the fractal dimensions \( d_f \) values that depend on \( \eta \): \( d_f = 1.3 \) for \( \eta = 1.0 \), \( d_f = 1.25 \) for \( \eta = 0.7 \), \( d_f = 1.15 \) for \( \eta = 0.5 \) and \( d_f = 1.1 \) for \( \eta = 0.2 \). Changing the elastic constants (i.e. the Lamé coefficients) just changes the opening angle of the crack.

The fact that using criterion II gives fractal structures is very different from what is seen in the scalar case of dielectric breakdown. it shows that neither noise nor long range time correlations are necessary to obtain fractals. The origin of fractality is the competition between a global stress perpendicular to the diagonal and a local stress that tends to continue a given straight crack due to tip instability. Again we see the important role of the interplay of different directions which is only possible in a truly vectorial model. The relevance of a short memory in criterion II indicates that there might be a relation between this case and the models that have been put forward for snow flakes [34].

In Fig. 7 we show a crack grown using criterion III for \( r = 0, f = 1, \eta = 1 \) and \( L = 118 \). The physical situation is similar to that seen in criterion II, only the fractal dimension is a little higher. This case can be directly compared to results obtained for DLA in the limit of infinite noise reduction [31, 35] where needles, not fractals are predicted.

In Fig. 8 we compare a deterministic crack with an experimental example of stress corrosion cracking in an alloy [36]. Due to the heuristic nature and simplicity of our model it makes no sense to compare numerical values of fractal dimensions. It seems also clear that the inhomogeneities of the medium are important during the growth of the experimental crack so that disorder should be included to modelize real cracks. For one-crack models a probabilistic approach similar to the one used for DLA [31] has been applied on the central force model [19, 21, 28]. the cracks obtained in this way seem to be fractal with a fractal dimension \( d_f \) that depends on the external boundary condition. Roughly speaking in two dimensions \( d_f \approx 1.6 \) has been found under dilation, i.e. if the system is pulled radially, \( d_f \approx 1.2 \) was obtained under uniaxial tension and \( d_f \approx 1.4 \) under shear. The
latter case, however, may just be a superposition of two clusters of dimension 1.2 [28]. Unfortunately the clusters from which these data were measured are by far not as large as state of the art DLA clusters. This is mainly due to the fact that no algorithm using random walkers has yet been implemented for elastic problems, although in principle the Lamé equations can be described by a stochastic process [37]. Progress in this direction would certainly be useful.

6. Scaling laws of the fracture of heterogeneous media

The models discussed in the last section produced one connected cluster, which in most cases is rather unrealistic. In fact, the possibility of having many cracks attracting or screening each other is crucial in fracturing if one is interested in a quantitative comparison with real forces or displacements. In the following we will consider a finite two-dimensional lattice $L \times L$ with periodic boundary conditions in the horizontal direction and fixed bus bars on top and bottom on which the external strain (elongation or shear) will be applied. Each bond is supposed to be ideally fragile; i.e. to have a linear elastic dependence between force $f$ and displacement $\delta$ with unit elastic constant up to a certain threshold force $f_c$ where it breaks (see Fig. 4) and in which the thresholds are randomly distributed [22].

For the beam model we introduce two random thresholds $t_i$ and $t_e$ and break the beam for which the $p$ of eq. (13) is largest. We distribute the thresholds according to eq. (14). If the bonds are electrical fuses [25] or springs which only can be subjected to a central force [32] the breaking criterion is to choose the bond with maximum value of $|i|/i$, where $i$ is the current or the central force respectively. The threshold is again randomly distributed and for the electrical case we will also consider a Weibull distribution as given in eq. (12).

Each time a bond is broken we monitor the external force $F$ and the external displacement $\lambda$ both averaged for a fixed number $n$ of bonds cut. The relation between the two gives the breaking characteristics of the entire system as shown in Fig. 9 for the beam model. We see that, as opposed to the single bond that was ideally elastic, the macroscopic characteristics are non-linear and after a maximum force $F_b$ has been applied, the system can still be elongated very much before getting disconnected, a regime experimentally only accessible if a displacement and not a force is imposed. After reaching the maximum, the breaking characteristics is subject to strong statistical fluctuations. For weak disorder ($m > 2$) the characteristics even bends back, i.e. both $F$ and $\lambda$ decrease; a behaviour called “class II” in rock mechanics which can only be measured with very stiff and quick, servo-controlled testing machines.

Before the maximum is reached, there is an initial regime with less statistical fluctuations and dominated by the disorder, i.e. it shrinks for decreasing disorder. In this regime Fig. 9 verifies the scaling law

$$F = L^{\lambda} \phi(\lambda L^{-\beta})$$

with $x \approx \beta \approx 0.75$. This law can be checked for all three models [33], all distributions $(0.8 \leq x \leq -1, 2 \leq m \leq 5)$ and for both, external extension and shear (in the elastic case) with exponents that agree with $3/4$ within 5 to 10%. For the same range of forces one finds for the number $n$ of bonds cut the scaling law

$$n = L^{\psi(\lambda L^{-\beta})}$$

with $\lambda \approx 1.7$ and the same universal range of validity as for eq. (18).

The number $n_b$ of bonds that have been cut when the force reaches the maximum scales again for most cases like $n_b \sim L^{1.7}$ as seen in Fig. 10a. Only when the disorder becomes very small, i.e. for $m = 5$ and 10, there seems to be a crossover to $n_b \sim L^k$ as expected. Force and displacement at the maximum do not seem to obey a power law relation at least for the small sizes considered.

Finally, after $n_f$ bonds are cut the system breaks apart altogether. Again a behaviour $n_f \sim L^{1.7}$ is reasonably well followed by the data except for small disorder where for $m = 5$ and $m = 10$ a crossover to the expected $n_f \sim L$ is observed (see Fig. 10b). Moreover, this finding seems univer-
sal with respect to the three models [33], the distribution of randomness and the external boundary conditions within our error bars (5 to 10% depending on the model). For the scalar model it can also be verified that the length of the largest crack, which causes the failure of the system, scales proportional to $L$ for all distributions considered.

The scaling relation for the number of bonds cut, valid during the whole breaking up to its end with an exponent of about 1.7, is not an expected result since Ref. [22] predicts $n \sim L/\ln L$ for the case $x = 0$ using the "dilute crack" approximation. This approximation is based on extreme value statistics [23, 38]: It is the longest crack that will determine the ultimate failure and its length is proportional to $\ln L$.

Let us analyze next the distribution $n(i)$ of local forces (shears and moments) and can also be considered or in the case of electrical fuses of local currents. The moments of this distribution are defined as $M_x = \Sigma_{bonds} i^n(i)$. In Fig. 11b $M_0$ and the quantities $m_x = (M_x/M_0)^{1/6}$ are plotted as function of $L$ for the scalar model at the point when the last bond is cut before the system breaks apart. We see that with varying $q$ the $m_x$ scale like $m_x \sim L^{1/6}$ with different exponents $y_q$. This is in sharp contrast to what happens if the same analysis is made to the $n(i)$ at the maximum of the breaking characteristics (see Fig. 11a). Here the $m_x$ fall on parallel straight lines for different $q$ and so all $y_q$ are the same (constant gap scaling).

The phenomenon of $y_q$ varying with $q$ shown in Fig. 11b is a manifestation of multifractality that has recently been observed in various contexts [39]. Another analysis, namely the investigation of the $f(x)$ spectrum [40], also leads to the conclusion that $n(i)$ is multifractal just before the last bond is cut also for the central force model [32] and the beam model [29].

Physically the multifractality means that the regions with highest variation in local strains, i.e. the regions that are finally responsible for rupture, lie on a fractal subset of the system. The fractal dimension of this subset depends on the strength of the local variations. In practical terms this means that the larger the system the more pronounced becomes the contrast between highly strained and practically unstrained regions. This effect only occurs just before the system breaks and not during the whole process as seen in Fig. 11a. Our data permit to quantify this statement. The appearance of multifractality is more astonishing if one considers that only a negligible number of bonds ($n \sim L^{1.7}$) has been cut, in contrast to the case of percolation where multifractality [39] only appears at the percolation threshold $p_c$, i.e. $n \sim p_c L^d$.

Local strains can be studied by photoelasticity and this might also be the best mean to verify the multifractal properties.

7. Conclusion

We have seen in this lecture how in fracture fractalts appear if formulated as a growth problem similar to viscous fingering. The vectorial nature of elasticity, however, generates already for deterministic models very complex patterns.

The role of disorder on fracture is very strong because fracture is dominated by the extreme moments of the distribution of local strain. In the case of quenched disorder we gave evidence for the existence of scaling laws in the breaking characteristics and multiscale just before the system breaks apart. A formulation of fracture as a critical phenomenon should therefore be possible.

The experimental reality of fracture is very rich and our description is at this time restricted to some sorts of brittle cracks. In order to compare the results from simulations to experimentally observed cracks [41] one must, however, overcome the numerical difficulties and get more reliable data.

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References

27. Meakin, P., in Ref. [4].