DAMAGE SPREADING: A RELATION BETWEEN MAGNETIC SYSTEMS AND CELLULAR AUTOMATA

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ABSTRACT. The concept of damage spreading in the context of Monte Carlo is introduced. Subtle differences between heat bath and Glauber dynamics give rise to totally different behavior in the damage spreading. In general damage spreading defines new dynamical phase transitions which in some cases can be related to usual thermal critical points. In spin-glass damage spreading can even generate multifractality.

1. INTRODUCTION

In the last years a new way of looking at Monte Carlo has led to the concept of damage spreading. The idea came from work that had been done on cellular automata and that ultimately originated from dynamical systems theory although there are some important differences.

Monte Carlo can be viewed as a dynamical process in phase space, i.e. starting from a given initial configuration one follows on a trajectory in phase space under the application of the Monte Carlo procedure. This trajectory will of course depend on the specific type of Monte Carlo (heat bath, Glauber, Metropolis, Kawasaki, Kadanoff-Swift etc) but also on the sequence of random numbers. The trajectories will always go towards equilibrium which means that the configurations will be visited by the trajectory with a probability proportional to the Boltzmann factors. Once in equilibrium the trajectories will stay there as assured by the detailed-balance condition.

The interesting question that we will ask now and which is the central issue of dynamical system theory is how much will a trajectory depend on the initial condition. More precisely: Suppose we make a small perturbation in the initial condition will the new trajectory be just slightly different or will it be totally different. The second case in which two initially close trajectories will become very fast very different is generically called chaos. The detailed definition of chaos may in some cases also include the speed with which the trajectories separate but in our context we do not want to make this distinction.

In order that the concept of closeness of trajectories be meaningful one must define what is a distance in phase space. Suppose we consider a system of Ising variables then a useful metric can be given by the "Hamming distance" or "damage"

\[ \Delta(t) = \frac{1}{2N} \sum_i |\sigma_i(t) - \mu_i(t)| \]  

(1)

where \( \{\sigma_i(t)\} \) and \( \{\mu_i(t)\} \) are the two (time-dependent) configurations in phase space and \( N \) is the number of sites, labelled by \( i \). This definition is certainly arbitrary and the results that we present in the following depend on this definition. Physically it just measures the fraction of sites for which the two configurations are different. The definition of Eq.(1) can easily be generalized to continuous variables or to variables off-site.

According to the discussion above one would then call a dynamical behaviour chaotic when in the thermodynamic limit \( \Delta(t) \) goes to a finite value for large times if \( \Delta(0) \to 0 \).
The opposite of chaotic is called frozen, i.e. when $\Delta(\infty) = 0$ if $\Delta(0) \to 0$. If one wants to get the limit $\Delta(0) \to 0$ properly in a numerical simulation one can use the following trick$^{1}$: Consider three configurations $\{\sigma_A\}, \{\sigma_B\}$ and $\{\sigma_C\}$ with $\Delta_{AB}(0) = \Delta_{BC}(0) = \frac{1}{2}\Delta_{AC}(0) = \varepsilon$ where $\varepsilon$ is a fixed small number and $\Delta_{AB}$ denotes the distance between $\{\sigma_A\}$ and $\{\sigma_B\}$. Then

$$\Delta(t) = \Delta_{AB}(t) + \Delta_{BC}(t) - \Delta_{AC}(t)$$

is a very good extrapolation to $\Delta(0) \to 0$.

It is now clear why the procedure that we will follow has been called "damage spreading". One creates at $t = 0$ an initial damage $\Delta(0)$ in one configuration (which gives a second configuration) and watches if the damage spreads (chaotic situation) or if it heals (frozen situation). This kind of spreading has been investigated before$^{10}$ in cellular automata to describe for instance the influence of perturbations in genetic regulatory systems.

The crucial idea in order to study damage spreading in Monte Carlo is therefore to take two configurations and to apply on them the same sequence of random numbers in the Monte Carlo algorithm. In this way one is taking the same dynamics for both configurations. To get statistically meaningful results one must then average over many initial configurations of equal initial distance and over many different sequences of random numbers. It is also important to note that, as in deterministic dynamical systems, once the two configurations become identical they will always stay identical.

2. NUMERICAL RESULTS FOR THE ISING MODEL

In order to see if Monte Carlo can generate chaotic behaviour in the sense outlined above Glauber dynamics was applied in Ref.1 on the two dimensional Ising model. Using Eq.(2) the distance of initially close equilibrium configurations was calculated after $10^4$ updates per site, i.e. after a long time. The result is shown in Fig.1 as a function of temperature. Let us remark that of course since the calculation was performed in a finite system of size $L$ the final distance would have vanished after a time of the order of $\exp(L^2)$ because eventually two uncorrelated trajectories in a finite phase space will always meet but the times considered in Fig.1 are much smaller than these "Poincaré" times. We see from Fig.1 that above a certain temperature $T_s$ the Glauber dynamics is chaotic while below $T_s$ it is frozen. The order parameter for this transition between chaotic and frozen is just the quantity plotted in Fig.1. The data seem to indicate that $T_s$ is very close if not identical to the critical temperature $T_c$ of the Ising model. It has not been possible up to now to determine the critical exponent $\beta$ of this transition because the statistical fluctuations are very strong.

In three dimensions using Glauber dynamics qualitatively the same picture emerges as in two dimensions only the temperature $T_s$ seems to be 4% below $T_c$ as found in Ref.3. Particularly interesting is the fact that one finds a transition line even in a homogeneous field$^{10}$ as shown in Fig.2. This indicates clearly that the dynamic transition between chaotic and frozen is not identical to the transition between the ferromagnetic and the paramagnetic phase. The transition line of Fig.2 does not seem to agree either with the percolation transition of minority spins$^{10}$. It is thus one of the open, challenging questions if this dynamical transition is related to any known property of the Ising model or if it is a totally novel phenomenon.
\( \Delta = 0 \) if \( \Delta(0) \rightarrow 0 \). If one wants to use the following \( \{ \sigma \} \) with \( \Delta_{AB}(0) = \Delta_{BC}(0) = \Delta_{AC}(t) \) denotes the distance between

\[
\Delta_{AC}(t)
\]

\( x \) has been called "damage spread" in one configuration (which gives preads (chaotic situation) or if it is investigated before in cellular patterns in genetic regulatory in Monte Carlo is therefore to one sequence of random numbers king the same dynamics for both one must then average over many many different sequences of random mimistic dynamical systems, once ya stay identical.

**MODEL**

Stochastic behaviour in the sense outlined a 0-dimensional Ising model. Using gurations was calculated after \( 10^4 \) s shown in Fig.1 as a function of e calculation was performed in a vanished after a time of the order stories in a finite phase space will small smaller than those "Poincaré" erature \( T_s \) the Glauber dynamics meter for this transition between The data seem to indicate that \( T_c \) \( \sim T_s \) of the Ising model. It has not one's \( \beta \) of this transition because iteratively the same picture emerges to be \( 4\% \) below \( T_s \) as found in Ref.3. nsion line even in a homogeneous the dynamic transition between the ferromagnetic and the not seem to agree either with the e of the open, challenging questions perty of the Ising model or if it is

**Fig.1**: Hamming distance \( 10^4 \) time step for \( \Delta(0) \rightarrow 0 \) using Glauber dynamics as a function of \( T/T_c \) for the two dimensional Ising model (from Ref.1).

**Fig.2**: Phase diagram in the field - temperature plane between the chaotic and the frozen phase for the three-dimensional Ising model obtained by Le Cué [4] for Glauber dynamics in a \( 10^3 \) lattice.
It was a lucky coincidence that independently the same questions were asked using a slightly different dynamics, namely heat bath. The authors calculated the fraction \( P(t) \) of the pairs of configurations that had not yet become identical after a time \( t \) and the average distance \( D(t) \) between only those not yet identical configurations, so that \( \Delta(t) = P(t) \cdot D(t) \). They started with not thermalized configurations and looked at different values of initial damage. Their result is shown in Fig. 3 after a time of 500 updates per site. One sees that the survival of damage depends on the initial damage (Fig. 3a) while when the configurations are different their distance does not depend on the initial distance (Fig. 3b). We also see that if the initial distance goes to zero the final distance is also going to vanish because of \( P(t) \). This is in agreement with the result found for Glauber dynamics in the ferromagnetic phase. In the paramagnetic phase, however, the result for heat bath is strikingly different from the Glauber dynamics because as seen from Fig. 3 heat bath does not show chaotic behaviour but on the contrary is in a frozen phase in which the final distance vanishes even if the initial distance was big. The opposite behaviour of the two dynamics is very surprising because normally it is thought that heat bath and Glauber dynamics are identical for the Ising model. We will investigate this question in the next section.

3. HEAT BATH vs GLAUBER DYNAMICS IN THE ISING MODEL

Let us consider variables \( \sigma_i = \pm 1 \) and define as \( h_i = \sum_{\mu} \sigma_{\mu} \) the local field acting on \( \sigma_i \) that comes from its nearest neighbors. Then one update in heat bath is given by setting the new value \( \sigma'_i \) of the spin to be \(+1\) with probability \( p \):

\[
P_i = \frac{e^{\beta h_i}}{1 + e^{\beta h_i}}
\]

(3)

On the computer this is implemented by choosing a random number \( z \in [0,1] \) and setting

\[
\sigma'_i = \text{sign}(p_i - z)
\]

(4)

In Glauber dynamics a spin is flipped with a probability

\[
p(\text{flip}) = \frac{e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}}
\]

(5)

where \( \Delta E \) is the difference between the energy of the would-be new configuration and of the old configuration. For the Ising model it is just \( \Delta E = 2\sigma_i h_i \). On the computer one implements the Glauber dynamics via

\[
\sigma'_i = -\sigma_i \text{ sign}(p(\text{flip}) - z)
\]

(6)

Using Eqs. (3) and (5) one can express \( p(\text{flip}) \) in terms of \( p_i \) by

\[
p(\text{flip}) = \begin{cases} 
1 - p_i & \text{if } \sigma_i = +1 \\
p_i & \text{if } \sigma_i = -1
\end{cases}
\]

and

\[
p(\text{not flip}) = \begin{cases} 
p_i & \text{if } \sigma_i = +1 \\
1 - p_i & \text{if } \sigma_i = -1
\end{cases}
\]

(7)
same questions were asked using authors\cite{8} calculated the fraction come identical after a time $t$ and t identical configurations, so that red configurations and looked at own in Fig.3 after a time of 500 ge depends on the initial damage heir distance does not depend on initial distance goes to zero the final in agreement with the result found the paramagnetic phase, however, Glauber dynamics because as seen but on the contrary is in a frozen tial distance was big. The opposite se normally it is thought that heat g model. We will investigate this

**THE ISING MODEL**

\[ h_i = \sum_{j=1}^{n} \sigma_j \]

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\[ (3) \]

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\[ (4) \]

ty

\[ (5) \]

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\[ (6) \]

of $p$, by

\[ (7) \]

\[ \sigma_i = +1 \]

if $\sigma_i = +1$

\[ \sigma_i = -1 \]

if $\sigma_i = -1$

\[ \Delta(0) \rightarrow 0 \] (a), $\Delta(0) = \frac{1}{2} (a)$ and $\Delta(0) = 1 (a)$ (a) fraction $p(t)$ of non-identical configurations and (b) distance $D(t)$ between two configurations provided they are still different.

![Figure 2](image)

It can easily be seen from Eq.(7) that for the Glauber dynamics $\sigma'_i$ is set +1 with probability $p$, and -1 with probability $1 - p$, just as in heat bath so that both dynamics have exactly the same probabilities.

Inserting Eq.(7) into Eq.(6) one finds how Glauber dynamics is implemented on the
computer:
\[
\begin{align*}
\sigma_i^1 &= \text{sign}(x - (1 - p_i)) \quad \text{if } \sigma_i = +1 \\
\sigma_i^2 &= \text{sign}(p_i - x) \quad \text{if } \sigma_i = -1
\end{align*}
\]  
(8)

This means that depending on the value of \(\sigma_i\) one uses the random number differently. So, if in one configuration the site \(\sigma_i = +1\) and in the other configuration it was \(-1\) but the nearest neighbors in both configurations are the same one the damage at site \(i\) will probably survive in Glauber dynamics while it will certainly heal in heat bath. This gives rise to the different behavior in damage spreading between the two methods.

4. RELATIONSHIP BETWEEN DAMAGE AND THERMODYNAMIC PROPERTIES

Damage can be related to correlation functions as has been shown in Ref.6. In the following we will briefly report on these relationships. We will consider Ising variables \(\sigma_i = \pm 1\) which can also be expressed as usual by \(\sigma_i = \frac{1}{2}(1 - \xi_i) = 0, 1\).

We want to discuss the damage between configurations \(\sigma_A\) and \(\sigma_B\) and in order to produce a small damage we will fit the spin at the origin of configuration \(\sigma_B\) to be always
\[
\sigma_0^B = -1
\]  
(9)

This condition is different from the ones considered before because now the damage is fixed constituting thus a source of damage. In principle two types of damage can be imagined at site \(i\) : damage \(+\) where \(\sigma_i^A = +1\) and \(\sigma_i^B = -1\) or damage \(-\) where \(\sigma_i^A = -1\) and \(\sigma_i^B = +1\). The probabilities of finding a certain type of damage at site \(i\) can then be expressed as:
\[
d_i^+ = \langle (1 - \sigma_i^A) \sigma_i^B \rangle \quad \text{and} \quad d_i^- = \langle \sigma_i^A (1 - \sigma_i^B) \rangle
\]  
(10)

where \(\langle \cdots \rangle\) denotes a time average. Let us define the difference between the damage:
\[
\Gamma_i = d_i^+ - d_i^- = \langle \sigma_i^A \rangle - \langle \sigma_i^B \rangle
\]  
(11)

where we have used Eq.(10).

We want to express the damage through thermodynamic quantities defined on an unconstrained system \(\{\xi_i\}\) with averages taken over many configurations, i.e. samples. So we translate condition (9) by a conditional probability and use ergodicity to go from time averages to thermal averages:
\[
\langle \sigma_i^A \rangle = \langle \xi_i \rangle \quad \text{and} \quad \langle \sigma_i^B \rangle = \frac{\langle \xi_i (1 - \xi_0) \rangle}{\langle 1 - \xi_0 \rangle} \tag{12}
\]

where \(\langle \cdots \rangle\) denotes a thermal average. Inserting this into Eq.(11) one finds
\[
\Gamma_i = \frac{\langle \xi_i \xi_0 \rangle - \langle \xi_i \rangle \langle \xi_0 \rangle}{1 - \langle \xi_0 \rangle} = \frac{C_0}{2(1 - m)} \tag{13}
\]

where
\[
C_0 = \langle \sigma_i \sigma_0 \rangle - \langle \sigma_0 \rangle^2 \quad \text{and} \quad m = \langle \sigma_0 \rangle \tag{14}
\]

are just the correlation function and the magnetization. Relation (13) gives us an equality of a certain combination of the damages with thermodynamical functions. In its derivation
we used ergodicity but did not make any assumptions on the dynamics, the random numbers or the type of interaction and it is therefore of a very general validity.

If one would have chosen another fixed result the would have changed slightly. If one would for instance fix $\sigma^B_0 = -1$ and $\sigma^A_0 = +1$ then one would find
$$\Gamma_i = \frac{C_{0i}}{1 - m^2}.$$  
If one considers variables with more degrees of freedom than Ising variables things can become more complicated but are still in principle feasible. As an example let us look at the Ashkin-Teller model where on each site one has two binary variables $\sigma_i = \pm 1$ and $\tau_i = \pm 1$ which follow a Hamiltonian per site
$$H_{ij} = -K (\sigma_i \sigma_j + \tau_i \tau_j) - 2L \sigma_i \tau_i \sigma_j \tau_j$$  
In this case there are twelve possible damages per site for which we label such that left means configuration $A$, right configuration $B$, top means $\sigma$ and bottom means $\tau$. Example $\pm \pm$ is $\sigma^A_i = +1$, $\tau^A_i = -1$, $\sigma^B_i = -1$ and $\tau^B_i = +1$. It can be shown\cite{7} that if one fixes $\sigma^B_i = -1$ and $\tau^B_i = +1$ or $\sigma^B_i = +1$ and $\tau^B_i = -1$ then
$$\Gamma_i = \frac{2 \langle \sigma_i \sigma_0 \rangle - \langle \sigma_i \rangle \langle \sigma_0 \rangle}{2 (1 - \langle \sigma_0 \rangle)}$$  
with
$$\Gamma_i = \begin{pmatrix} + & + & + & + & + & + & + & + & - & - & - & - \\
+ & + & + & + & + & + & + & + & + & + & + & + \end{pmatrix}$$

and if one fixes $\sigma^B_i = -1$ then
$$\bar{\Gamma}_i = \frac{2 \langle \sigma_i \sigma_0 \rangle - \langle \sigma_i \rangle \langle \sigma_0 \rangle}{2 (1 - \langle \sigma_0 \rangle)}$$

with
$$\bar{\Gamma}_i = \begin{pmatrix} + & + & + & + & + & + & + & + & - & - & - & - \\
+ & + & + & + & + & + & + & + & + & + & + & + \end{pmatrix}$$

This shows that both types of correlation functions that one has in the Ashkin-Teller model can be expressed as a combination of damages.

The damage for which we have presented numerical data in the preceding section was not the quantity $\Gamma_i$ but it was the sum of all the damages:
$$\Delta = \sum_i \Delta_i \quad \text{with} \quad \Delta_i = \sigma_i \tau_i$$

In order to express these quantities in terms of thermodynamic quantities it is necessary to make some assumptions. Let us therefore restrict ourselves now to ferromagnetic interactions, heat bath dynamics and the use of the same random numbers for both
configurations. We consider again only Ising variables and fix the damage as in Eq. (9), i.e. $\sigma_i^R = -1$. Since at $t = 0$ the only damage one has is the one of type $++$ at the origin we have

$$p^A_i \geq p^R_i \quad \forall i$$

(21)

where $p^A_i$ is the value defined in Eq. (3) for configuration $A$. Suppose one would try to create a damage of type $-+$ at site $i$. Then one would need, in order to produce $\sigma_i^A = -1$, a random number $x$ which fulfills $x \geq p^A_i$ according to heat bath. Since one is using the same random number for configuration $B$ this means using Eq. (21) that $x \geq p^B_i$ and therefore $\sigma_i^B = -1$. It is therefore impossible to create a damage of type $-+$ and therefore Eq. (21) will be valid also at the next time step. By induction one can conclude now that Eq. (21) will always be valid and that a damage of type $-+$ cannot be created under the conditions that we had imposed. Consequently we have proved that $d^{-+} = 0$ and it follows for the damage

$$\Delta_i = \frac{C_{0i}}{2(1 - m)} \quad \text{and} \quad \Delta = \frac{X}{2(1 - m)}$$

(22)

where $\chi = \sum_i C_{0i}$ is the susceptibility.

In Figs. 4 and 5 we see how the two exact relations of Eq. (22) are realized numerically for the 2d Ising model. In Fig. 4 we see the susceptibility obtained from Eq. (22) and obtained from the fluctuations of the magnetization as usual taken in a small system and the data agree very nicely. In Fig. 5 we see the correlation function obtained via Eq. (22) (circles) and in the usual way (triangles) using for both methods roughly the same computational effort. One sees that in the usual method once the values are less than about $10^{-3}$ the statistical noise takes over and the curve flattens. On the other hand, using Eq. (22) one gets to much smaller values without feeling substantial noise. The reason why the use of damage is superior numerically comes from the fact that this method looks just at the difference between two configurations subjected to the same thermal noise so that this noise is effectively cancelled. A similar fact was already pointed out for continuous systems by Parisi[4].

5. DAMAGE CLUSTERS

The damage that was fixed at the origin at time $t = 0$ acts as a source from which constantly damage spreads away. At a given instant one can look at all the sites that are damaged and one will find a cloud or cluster of sites around the origin. These clusters fluctuate in size and shape and are not necessarily connected. In Figs. 6a and b we see two of these clusters for the two dimensional Ising model at $T_c$ which are just 38 time steps apart. Since heat bath was used these clusters represent according to Eq. (22) the fluctuation of the magnetization due to the application of a local field at the origin. In Fig. 6c we see what happens if a damage $\sigma_i^R = -1$ is fixed all along the boundary of the system. Then clusters of the type shown in Figs. 6a and b overlap and one finds fluctuations that seem to be of all sizes. Using the equivalence of the Ising model to a lattice gas one can interprete these fluctuations as the fluctuations in local density which have actually been measured in a recent experiment[9].

Using Eq. (22) it is actually possible to calculate the dependence between radius and number of sites for the damage clusters. One finds that at the critical point the clusters are fractal, that means that if $L$ is the size of the smallest box into which the cluster fits and $s$ is the number of sites in the cluster (averaging over all clusters) one has a relation

$$s \propto L^{d_f}$$

(23)
and fix the damage as in Eq.(9), is the one of type $+-$ at the origin

$$z = \frac{X}{2(1-m)}$$  (22)

A. Suppose one would try to seed, in order to produce $a^a = -1$, a
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$$s = \int_{B} \Delta_a \, d^2r \propto \int_{0}^{L} G(r) \, r^{d-1} \, dr$$

where $G(r) \propto r^{d-1} \Sigma_{i=1} C_i$, and
using the fact that at $T_c$ one has $G(r) \propto r^{3-d-a}$ one obtains $d_f = 2 - \eta$. This can then
be transformed into $d_f = d - 2 \beta/\nu$ using the hyperscaling relation where $\beta$ and $\nu$ are
the critical exponents for the magnetization and for the correlation length. The above
argument is only valid if one really measures directly the damage $\Delta_i$, i.e. if one has the
condition that the origin be permanently damaged. We note that in percolation theory a
similar situation arises but there one is generally interested in the the mass of the largest
cluster so that explicitly the probability of being on the largest cluster has to be put in
by hand which gives an additional factor of $\beta/\nu$ in the exponent.

Since one does not expect several length scales in the problem one can also replace
in Eq.(23) $L$ by the radius of gyration $R$. In Fig.7 we see results for the numerical determination of $d_f$ for the 2d Ising model at $T_c$ and an agreement with $d_f = \frac{5}{4}$ is reasonably
good.

It is also possible to make similar arguments for the case when the damage is fixed

Fig.4: Susceptibility $\chi(\bullet)$ and $2\Delta(1-m)(\Delta)$ as function of temperature from 30 systems
of size 10 x 10 (taken from Ref.6) for the 2d Ising model.

Using Eq.(22), i.e. $s = \int_{\Omega} \Delta_a d^2r \propto \int_{0}^{L} G(r) r^{d-1} dr$ where $G(r) \propto r^{d-1} \Sigma_{i=1} C_i$, and
using the fact that at $T_c$ one has $G(r) \propto r^{3-d-n}$ one obtains $d_f = 2 - \eta$. This can then
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Fig. 5: Correlation function \( G(r) = \sum_{n=1}^{\infty} C_{vn}(\Delta \text{ for } T = 2.6 \text{ and } \triangle \text{ for } T = 3.0) \) and 
\( 2\Delta(r)(1-m)(\bullet \text{ for } T = 2.6 \text{ and } o \text{ for } T = 3.0) \) where \( \Delta = \sum_{n=1}^{\infty} \Delta_n \) as a function of \( r \) in a semi-log plot. The data come from 10 systems of size 40 \times 40 and were taken from Ref. 8 for a 2d Ising model.

on the boundary as shown in Fig. 6c. There the density of damaged sites in the center of the box decreases with a power law in the size of the box and consequently one finds again a fractal dimension as shown also in Fig. 7.

The fractal dimension of the clusters can also be measured by the touching method that has been widely used in cellular automata. One lets the damage spread until it touches the boundary of the system. The number of sites \( S \) of the touching cluster that are damaged at this time scales like \( S \propto L^{D_f} \) with the size \( L \) of the system, where \( D_f \) has the same value as \( d_f \) of Eq. (23) plus \( \beta/\nu \) which takes into account the probability of finding a touching cluster. In Fig. 8 we see the result for the three dimensional Ising model and one finds numerically 1.95 which is not in good agreement with \( D_f = d - \beta/\nu \). The reasons for the numerical discrepancy are probably strong corrections to scaling and the fact that the system sizes in Fig. 8 are quite small. Fig. 8 was obtained by the condition
Fig. 6: Damage clusters of the 8d Ising model at $T_c$. In (a) and (b) the damage is fixed to be $q^2 = -1$ at the origin and the size of the system is $L = 60$. In (c) the same kind of damage is fixed all along the boundary and $L = 100$. Heat bath and the same random number sequence were applied on both configurations (taken from Ref.6).

damage touches all the sides of the box, if instead one demands that the damage touch only one side of the box the same result is found. Let us note that if one does not fix a damaged site and if one uses Glauber dynamics instead of heat bath the result changes and one seems to find compact clusters (Ref.1).

6. DAMAGE IN SPIN GLASSES

Numerical work on spin glasses has been challenging and frustrating in the past so that any new method that gives some hope of improving the results should be tested. The application of the ideas of damage spreading has been done in Ref.5 using heat bath and they have been very interesting. The results are shown in Fig.9 which in their spirit are analogous to the data of the pure Ising model shown in Fig.8. In three dimensions one believes that about $T_{SG} = 1.2$ their is a transition to a spin glass phase for the spin glass with randomly chosen coupling constants of $+1$ or $-1$. At $T_G = 4.5$, the critical temperature of the pure Ising model, one believes that the paramagnetic phase changes into a so-called Griffiths phase in which correlations decay a little shorter than exponential, but this phase is very difficult to detect or do discern from the paramagnetic behaviour.

From Fig.9 one sees indeed two characteristic temperatures not too far from $T_{SG}$ and $T_G$ which could be interpreted as separating three phases. As opposed to the ferromagnetic phase of Fig.3 both low temperature phases here are chaotic because the probability
Fig. 7: Log-log plot of the number \( s \) of damaged sites against the radius \( R \) of the cluster (△) for the 2d Ising model at \( T_c \) with \( a_B^0 = -1 \) fixed. We also show the total number \( D^* \) of damaged sites if the damage is fixed on the boundary of a system of size \( L \) as a function of \( L \) (○). (Taken from Ref. 6).

of two configurations to be still different after some time is not zero for times large but much shorter than the Poincaré time. In the low temperature phase the final distance does depend on the initial distance while in the intermediate phase the final distance is independent of the final distance. One can indeed argue that this behaviour reflects the phase-space structure of the spin-glass phase and of the Griffiths phase. Unfortunately one finds qualitatively in two dimensions\(^{10}\) the same picture as the one seen in Fig. 9 but it is believed that there is no spin glass phase in two dimensions. This can be due to the fact that all these data are extremely far from an equilibrium-like state and must be treated with much care or it could be that the interpretation of the dynamical, chaotic phases of Fig. 9 cannot be related directly to the thermodynamic phases.

There exists however a way to single out \( T_G \) of the three-dimensional spin glass. During a certain interval of time a site will be damaged and healed again several times.
against the radius \( R \) of the cluster \( d \). We also show the total number of entries of a system of size \( L \) as a function of \( L \). \( \sigma^D_0 = -1 \) was fixed and heat bath dynamics was used.

One can look for each site \( i \) which is the number of times \( f_i \) that it is damaged again during a fixed, long time interval. Next one can look at the distribution \( P(f) \), i.e. the number of sites that have a certain frequency \( f \) and analyze the moments of this distribution:

\[
M_q = \sum f^q P(f) \quad q = 0, \pm 1, \cdots
\]

and normalize them:

\[
m_q = (M_q/M_0)^{1/q}
\]

These moments will scale with the size of the system \( L \) like

\[
m_q \propto L^{q*}
\]
Fig. 9: Damage in the three dimensional ± spin glass after 500 time steps as a function of temperature for \( L = 12 \) taken from Ref. 5 for various initial damages: \( \Delta(0) = 0 \) (c), \( \Delta(0) = \frac{1}{2} \) (a) and \( \Delta(0) = 1 \) (Δ); (c) fraction \( P(t) \) of non-identical configurations and (b) distance \( D(t) \) between two configurations provided they are still different.

Usually the distribution is self averaging and scales such that all \( x_q \) (with \( q \neq 0 \)) are the same. But in specific cases, when the distribution has a particularly long tail it can show what has been called multifractality which means that the \( x_q \) vary with \( q \) and one has therefore an infinity of different scaling exponents.

In Ref. 11 the moments of Eq. (25) have been calculated for various models. For the three dimensional spin glass it has been found that the distribution is multifractal at \( T_G \) as can be seen from Fig. 10a. This is in marked contrast to what has been found at \( T_{SG} \) for the 3d spin glass, at \( T_G \) or \( T_{SG} \) for the 2d spin glass or at the critical temperature \( T_c \) of the 3d Ising model because in all these cases the lines for the different moments in the log-log plot are parallel to each other which means that \( x_q \) is the same for all \( q \). As an example we show in Fig. 10b the data for the 3d Ising model.

We can conclude that the temperature \( T_G = 4.5 \) of the 3d ± spin glass is particular
after 500 time steps as a function of initial damages: $\Delta(0) \rightarrow 0$ ( ), non-identical configurations and ( ) are still different.

Each that all $x_q$ (with $q \neq 0$) are the same as long tail it can show the $x_q$ vary with $q$ and one has calculated for various models. For the $x_1$ distribution is multifractal at $T_G$ set to what has been found at $T_{SG}$ or at the critical temperature $T_c$ set for the different moments in the hat $x_q$ is the same for all $q$. As an example of the 3d ± spin glass is particular

Fig.10 : Log-log plot of the moments of the distribution $P(j)$ of the redamaging of sites as a function of the system size $L$ for (a) the 3d ± spin glass at $T_G$ (b) the standard 3d Ising model at $T_c$. The moments are: $M_0$ ( ), $M_1$ ( ), $M_2$ ( ), $M_3$ ( ) and $M_4$ ( ). Taken from Ref.11.

because there the distribution of the redamaging frequency is multifractal while in the usual case, particularly at $T_G$ in the 2d ± spin glass one has a simple one-exponent scaling behaviour. This feature might be useful to single out models that show spin glass behaviour.

7. CONCLUSION

We have seen that the notion of damage spreading that has been widely used for cellular automata can be applied to Monte Carlo. One finds dynamical phase transitions which only in specific cases like heat bath can be related to well-known thermal critical points. Much has to be understood yet about these dynamical transitions in the general case.

Up to now damage spreading techniques seem to be useful for two purposes: more precise calculation of correlation functions and eventually dynamical information about
spin glasses.

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8. REFERENCES