FRACTURE PATTERNS AND SCALING LAWS

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Fracture is determined by the growth of cracks and can be described as a moving boundary problem similar to dielectric breakdown or viscous fingering. The vectorial nature of the elastic medium allows, however, for a much larger variety of boundary conditions and growth rules than in the scalar electrostatic analogue giving rise to a number of new effects. One can model this problem on a square lattice and it is found that the patterns of cracks can be fractal even without including noise due only to the interplay of anisotropy and memory. The shapes of the cracks can be compared to the ones found experimentally for stress corrosion, an effect that can be mimicked by a damage memory on the surface of the crack. A quantity of experimental interest is the breaking characteristics (force vs. displacement). In a model with quenched disorder and the possibility that several cracks be present in the system one finds scaling laws in the system size for the breaking characteristics. Just before the system breaks fully apart the distribution of local strain is multifractal. This means that the volume fraction responsible for the final cracking goes to zero in the thermodynamic limit. The scaling and multiscaling laws seem to be quite universal.

1. Introduction

The formation of cracks in an elastic medium and its subsequent failure is a question of technological importance but also an interesting theoretical problem. The mechanisms leading to fracture are highly material dependent and have been studied quite extensively [1]. Despite the diversity of experimental situations, one hopes to find generic features due to the underlying instabilities and their interplay with noise, anisotropy or memory effects. Recently two new approaches in this direction have been proposed [2, 3], one inspired from random resistor networks [4] and another using DLA [5] as a guideline [6, 7].

Here results are presented for two different models for fracture, firstly a model for the growth of one single, connected, fractal crack and secondly a model for the breakdown on a solid with quenched disorder due to the formation and propagation of many cracks. In the next section we describe fracture as a moving boundary problem and present a possible discretization of the equations. In section 3 we study deterministic rules which can generate a fractal crack and in section 4 we discuss some scaling laws that have been observed numerically in systems in which the breaking thresholds are randomly distributed.
2. Modelization of fracture

An elastic medium is usually described by the field of displacement vectors $u$ obeying an equation of motion which in the case of symmetric elasticity is the Lamé equation [8]

$$(\lambda + \mu)\nabla(\nabla u) + \mu \nabla^2 u = 0,$$  \hspace{1cm} (1)$$

where $\lambda$ and $\mu$ are the Lamé coefficients. On the external boundary an imposed displacement is fixed. Suppose that in this medium one has already a crack and one wants to study how this crack grows. Then on the surface of the crack one will have the boundary condition that the stress normal to the surface of the crack is zero and the crack will grow in a direction perpendicular to the surface at the point where the strain parallel to the surface is largest. The detailed growth law depends on the microscopic mechanism like how the elastic energy is transported away from the growing tip. It is reasonable and quite general to assume a normal growth velocity $v_n$ of the form

$$v_n \propto ((\partial_\| u_\|)^2 + r \partial_\perp^2 u_\perp)^\eta,$$  \hspace{1cm} (2)$$

where $r$ and $\eta$ are material-dependent parameters. For $\eta = 1$ this growth law is inspired by the von Mises yielding criterion [1], but it is not possible to derive it from first principles. Physically $r$ is the affinity of the breaking process to the bending mode (second term on the rhs of eq. (2)) as compared to cleavage (first term on the rhs of eq. (2)). The time derivatives in eq. (1), like the inertia terms, are neglected. We do not consider plasticity or non-linear elasticity.

In the above approach we usually assume that the relaxation of the local strain due to the growth of the crack is much faster than the velocity of the crack which means that we do not consider viscoelastic effects. One can, however, easily introduce a finite relaxation time of the elastic system with respect to the typical time to grow the crack by a given length if one uses numerical relaxation techniques and in this way viscoelasticity can be effectively implemented.

The elastic equations can be discretized on a square lattice by using the beam model [9]. In fact the beam model is even richer than eq. (1) because it allows for local rotations in the medium, i.e. discretizes the equations of motion of asymmetric elasticity [10]. The discretization introduces anisotropy and a cutoff at small length scales, two physical effects that are very often present in real materials.
A detailed description of the beam model is given in ref. [2]. We implement eq. (3) as follows: For each beam that is eligible for being broken one calculates the quantity $p$ defined by

$$p = (f^2 + r \max(|m_1|, |m_2|)^\alpha),$$

(3)

where $f$ is the traction (and/or compression) force applied on the beam and $m_1$ and $m_2$ are the moments that are acting at the two ends of the beam; this $p$ determines according to eq. (2) if the beam will be broken. Each time a beam is broken the shape of the crack and consequently the boundary condition of the equation of motion changes and one has to solve the discretized equation again if one wants to know which beam to break next.

3. Deterministic growth of a fractal crack

Some progress has been made by modelling the growth of a single, connected crack. It was found in the numerical simulations of central-force media with a breaking probability proportional to the elongation of the springs that the cracks formed are fractal [6, 7]. The fractal dimension of these cracks seems to depend strongly on the type of external force that is applied (uniaxial tension, shear, uniform dilatation) but since only very small cracks can be grown, precise statements are difficult to make. Here, we will investigate some of the origins of this fractal behavior and obtain a much better accuracy by considering deterministic models.

We consider a finite square lattice of linear size $L$, with periodic boundary conditions in the horizontal direction. On the top and on the bottom we impose an external shear. We remove one beam in the center of the lattice which represents the initial microcrack. Next we consider six nearest-neighbor beams of this broken beam. These include two beams that are parallel to the broken beam and four perpendicular beams that touch a common site with the broken beam. This choice of nearest-neighbors comes from the fact that the actual crack consists of bonds that are dual to the set of broken beams [2]. Other connectivity conditions have also been used [7]. The Lamé equation is solved by a conjugate gradient method [11] to very high precision ($10^{-20}$) and the $p$’s of eq. (3) are calculated for each of the nearest-neighbor beams. We set $p = 0$ for a beam that is not a nearest-neighbor to the crack. Now various criteria for breaking are possible: I) One breaks the beam with the largest value of $p$. II) One breaks the beam for which $q_0 = p + f_0 p_{-1}$ is largest, where $p_{-1}$ is the value of $p$ that this beam had before the previous beam was broken; $f_0$ is a memory factor. III) On each beam of the lattice we put a counter $c$
which is set to zero in the very beginning. Each time one has obtained the \( p \)'s one calculates \( \alpha = (1 - c)/p \) and breaks the beam which has the smallest \( \alpha \), namely \( \alpha_{\text{min}} \). After the beam has been broken each counter \( c \) is set to \( c = \alpha_{\text{min}} p + fc_{-1} \), where \( c_{-1} \) is the value the counter had before the breaking and \( f \) is another memory factor.

All three breaking criteria described above are deterministic. Criterion III corresponds for \( f = 1 \) to the limit of infinite noise reduction [12]. Noise reduction was invented to reduce statistical noise in DLA simulations and has also been applied recently to central-force breaking [13]. What noise reduction certainly does is to introduce a memory effect with long range time correlations. Physically the three breaking criteria defined above correspond to three different situations. Criterion I describes ideally brittle and fast rupture. Criterion II contains a short time memory one would expect in cracks that produce strong local deformations at the tip of the crack as happens in most realistic situations and where this local damage does not heal much faster than the speed of the crack. Criterion III could be applied to situations to stress corrosion or static fatigue. The memory factors \( f_0 \) and \( f \) measure the strength of these time correlations. In criterion III the limit \( f \to 0 \) gives criterion II with \( f_0 = 1 \), and in criterion II the limit \( f_0 \to 0 \) gives criterion I.

Let us next discuss the results that one finds [14] for the above model. If one breaks according to criterion I, as shown in fig. 1 for \( L = 50 \) and \( r = 0.28 \), the cleavage tends to have the crack grow in the diagonal direction while the bending mode favours a horizontal rupture. The competition between these two effects can lead to complex branched structures. The exact shape of these cracks strongly depends on \( r \) and the system size. For any finite \( r \) the horizontal rupture will eventually win if the system is large enough while for \( r = 0 \) one

![Fig. 1. Crack grown in a 50 \( \times \) 50 system if an external shear is applied. Beams break under traction with an affinity of \( r = 0.28 \) in the bending mode. Only the bond with the largest value of \( q \) breaks (criterion I). The first broken beam was vertical (taken from ref. [14]).](image)


obtains diagonal cracks with eventual kinks. For this reason the cracks will not be fractal. In the analogous scalar model (i.e. DLA), however, only straight lines will be formed in criterion I; the different behaviour here is due to the fact that competing directions are possible in a vectorial model.

Let us now consider cracks grown using criterion II. We see in fig. 2 and fig. 3 cracks with \( r = 0 \) and \( \eta = 1.0 \) obtained in a system of size \( L = 118 \). Over four hours on one CrayXMP processor were needed to generate each of these structures. The crack in fig. 2 is obtained by breaking only through tension while in fig. 3 a beam can also break under compression. In fig. 3 we show the whole crack which has a four-fold symmetry because it is grown deterministically, i.e. no random numbers are used and the result is independent of the

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**Fig. 2.** Upper half of a crack grown in a \( 118 \times 118 \) system under external shear using criterion II with \( f_0 = 1, \eta = 1.0, r = 0 \) (breaking through traction only) and vertical first broken beam.
roundoff errors of the computer. In fig. 2 we only show the upper part of the crack, the lower part being reflection symmetric. The very slight curvature of the crack is a finite size effect of the lattice which is, however, such a weak effect that its influence cannot be noticed quantitatively, for instance, in the value of the fractal dimension that we will discuss next.

If we count the number of broken beams inside a box of length \( l \) around the first broken beam and plot it as a function of \( l \) in a log–log plot ("sand box method") we find lines with slopes larger than unity which means that the cracks are fractals. In system sizes of \( L = 118 \) we find for the fractal dimensions \( d_t \) values that depend on \( \eta \): \( d_t = 1.3 \) for \( \eta = 1.0 \), \( d_t = 1.25 \) for \( \eta = 0.7 \), \( d_t = 1.15 \) for \( \eta = 0.5 \) and \( d_t = 1.1 \) for \( \eta = 0.2 \). The structures are self-similar around the origin and are probably directed fractals [15]. Changing the elastic constants (i.e. the Lamé coefficients) just changes the opening angle of the crack. If in eq. (3) one uses an exponential instead of a power law, the structures seem to be dense [15].

The effect that using criterion II gives fractal structures is novel and very distinct from what is seen in the scalar case of DLA. It is shown that neither noise nor long range time correlations are necessary to obtain fractal breakdown. The origin of fractality is the competition between a global stress perpendicular to the diagonal and a local stress that tends to continue a given straight crack due to tip instability. Again we see the important role of the interplay of different directions which is only possible in a truly vectorial model. The relevance of a short memory in criterion II indicates that there
might be a relation between this case and the models that have been put forward for snow flakes [16].

In fig. 4 we show a crack grown using criterion III for \( r = 0, \eta = 1 \) and \( L = 118 \). The physical situation is similar to that seen in criterion II, only the fractal dimension is higher. This case can be directly compared to results obtained for DLA in the limit of infinite noise reduction [13, 17] where needles, not fractals, are predicted.

In fig. 5 we compare a deterministic crack with an experimental example of stress corrosion cracking in an alloy [18]. Due to the heuristic nature and simplicity of our model it makes no sense to compare the numerical values of fractal dimensions. It seems also clear that the inhomogeneities of the medium in the experimental crack are important. The vague similarity that one can see between the two patterns in fig. 5 seems to indicate that the effects found in our model may explain to a certain degree the branching behaviour of experimental cracks.

Fig. 4. Upper half of a crack grown in a \( 118 \times 118 \) system using criterion III with \( f = 1, \eta = 1, r = 0 \) and vertical first broken beam when the beams break under traction only.
Fig. 5. Numerical and experimental cracks. The upper shape is the upper half of a crack grown in a 118 × 118 system under external shear with \( r = 0 \) and a vertical initially broken beam using criterion II with \( f_0 = 1 \) and \( \eta = 0.2 \). The lower picture shows the morphology of cracking in Ti–11.5 Mo–6 Zr–4.5 Sn aged 100 h at 750 K and tested in 0.6 M LiCl in methanol at \(-500 \text{ mN}\) under increasing stress intensity (taken from ref. [18]).

4. Scaling laws of the fracture of heterogeneous media

The model discussed in the last section contained no disorder, which is a rather unrealistic simplification. In fact, the role of disorder is crucial in fracture if one is interested in a quantitative comparison with real forces or displacements. In the following we will consider a finite two-dimensional lattice \( L \times L \) with periodic boundary conditions in the horizontal direction and fixed bus bars on the top and bottom on which the external strain (elongation or shear) will be applied. Each bond is supposed to be ideally fragile; i.e. to have
a linear elastic dependence between force \( f \) and displacement \( \delta \) with unit elastic constant up to a certain threshold force \( f_c \) where it breaks (see insert of fig. 6). We will discuss a model [19] in which the thresholds are randomly distributed according to some probability distribution \( P(f) \). Once a force beyond \( f_c \) is applied to a bond, it is irreversibly removed from the system. As the external strain is increased one can watch bonds breaking one by one until the system falls apart altogether.

For the beam model one has to introduce actually two random thresholds, \( f_c \) and \( m_c \), and break a beam if [20]

\[
(f/f_c)^2 + \max(|m_1|, |m_2|)/m_c \geq 1,
\]

which is equivalent to eq. (3). We distribute the thresholds according to \( P(f_c) = (1-x)f_c^{-x} \) with \( 0 < f_c \leq 1 \) and \( P(m_c) = (1-x)r^{-1}m_c^{-x} \) with \( 0 < m_c \leq r \), where \( x \) and \( r \) are parameters. It is also possible to consider models in which the bonds are not beams but simpler objects, like electrical resistors [21] or springs which can only be subjected to a central force [22]. In these cases the breaking criterion is \( |i|/i_c \geq 1 \) where \( i \) is the current or the central force, respectively. The threshold is again randomly distributed and for the electrical

![Graph](Fig. 6. Breaking characteristics of the beam model with \( x = 0.5 \), \( r = 1 \) with both axis scaled by \( L^{-3/4} \) for different sizes \( L \). The data have been smoothened to reduce statistical fluctuations [6]. The insert shows the characteristics of a single beam (taken from ref. [24]).)
case we will also consider a Weibull distribution, namely \( P(i_c) \propto i_c^{m-1} e^{-\left(i_c / i_0\right)^m} \) where \( m \) and \( i_0 \) are parameters.

Each time a bond is broken let us monitor the external force \( F \) and the external displacement \( \lambda \) both averaged over the fixed number \( n \) of bonds cut. The relation between the two gives the breaking characteristics of the entire system as shown in fig. 6 for the beam model. We see that, opposed to the single bond that was ideally brittle, the macroscopic characteristics is ‘ductile’ by which we mean that, after a maximum force \( F_b \) has been applied, the system can still be elongated very much before getting disconnected, a regime experimentally accessible only if a displacement and not a force is imposed. After reaching the maximum, the breaking characteristics is subject to strong statistical fluctuations and for weak disorder \( (m > 2) \) it seems to bend back; i.e. both \( F \) and \( \lambda \) decrease. Before the maximum is reached, there is an initial regime with less statistical fluctuations and dominated by the disorder, so that it shrinks for decreasing disorder. In this regime fig. 6 verifies the scaling law

\[
F = L^\alpha \phi(\lambda L^{-\beta}),
\]

with \( \alpha \approx \beta \approx 0.75 \). This law can be checked for all three models [23], all distributions \( (0.8 \leq x \leq -1, 2 \leq m \leq 5) \) and for both external extension and shear (in the elastic case) with exponents that agree with \( 3/4 \) within \( 5 \) to \( 10\% \). For the same range of forces one finds the scaling law for the number of bonds cut

\[
n = L^\gamma \psi(\lambda L^{-\beta}),
\]

with \( \gamma \approx 1.7 \) and the same universal range of validity as for eq. (5).

The number \( n_b \) of bonds that have been cut when the force reaches the maximum scales again for most cases like \( n_b \sim L^{1.7} \) as seen in fig. 7(a). Only when the disorder becomes very small, i.e. for \( m = 5 \) and \( 10 \), there seems to be a crossover to \( n_b \sim L^{0.9} \) as expected. Force and displacement at the maximum do not seem to obey a power law relation at least for the small sizes considered.

Finally, after \( n \) bonds are cut the system breaks apart altogether. Again a behaviour \( n_t \sim L^{1.7} \) is reasonably well followed by the data except for small disorder where for \( m = 5 \) and \( m = 10 \) a crossover to the expected \( n_t \sim L \) is observed (see fig. 7(b)). For the scalar model it can also be verified that the length of the largest crack, which causes the failure of the system, scales proportional to \( L \) for all distributions considered.

The scaling relation for the number of bonds cut, valid during the whole breaking up to its end with an exponential of about 1.7, is an unexpected result.
Fig. 7. Number of bonds cut when (a) the maximum force is reached and (b) when the system falls apart in a log–log plot against $L$ for the scalar model with $x = 0$ (□), $x = 0.5$ (○), $m = 2$ (●), $m = 5$ (×) and $m = 10$ (○) and the beam model ($r = 1$) for an external elongation with $x = 0$ (+), $x = 0.5$ (▲), $x = -1$ (▲) and for an external shear with $x = 0$ (▼). The full lines are guides to the eye, of slope 1.7 (taken from ref. [24]).

since ref. [4] predicts $n \sim L$ for the case $x = 0$. Moreover, this finding is universal with respect to the three models [23], the distribution of randomness and the external boundary conditions within our error bars (5 to 10% depending on the model).

Let us analyze next the distribution $n(i)$ of local currents (local forces, shears and moments in the elastic case) at the point when the last bond is cut before the system breaks apart. The moments of this distribution are defined as $M_q = \sum_{\text{bonds}} i^q n(i)$. In fig. 8(b) $M_0$ and the quantities $m_q = (M_q/M_0)^{1/q}$ are plotted as functions of $L$ for the scalar model. We see that with varying $q$ the $m_q$ scale like $m_q \sim L^{y_q}$ with different exponents $y_q$. This is in sharp contrast to what happens if the same analysis is applied to the $n(i)$ at the maximum of the breaking characteristics (see fig. 8(a)). Here the $m_q$ fall on parallel straight lines for different $q$, so all $y_q$ are the same (constant gap scaling).

The phenomenon of $y_q$ varying with $q$ shown in fig. 8(b) is a manifestation of multifractality that has recently been observed in various contexts [24]. Another analysis, namely the investigation of the $f(\alpha)$ spectrum [25], also leads to the conclusion that $n(i)$ is multifractal just before the last bond is cut also for the central force model [22] and the beam model [20].

Physically the multifractality means that the regions with highest variation in local strains, i.e. the regions that are finally responsible for rupture, lie on a
fractal subset of the system. The fractal dimension of this subset depends on the strength of the local variations. In practical terms this means that the larger the system is the more pronounced the contrast between highly strained and practically unstrained regions becomes. This effect only occurs just before the system breaks and not during the whole process as seen in fig. 8(a). Our data permit to quantify this statement. The appearance of multifractality is more astonishing if one considers that only a negligible number of bonds \( n \sim L^{1.7} \) have been cut, in contrast to the case of percolation where multifractality [24] only appears at the percolation threshold \( p_c \) (i.e. \( n = p_c L^2 \)). Local strains can be studied by photoelasticity and this might also be the best mean to verify the multifractal properties.

5. Conclusion

In conclusion, we have discussed some selected problems of crack propagation on finite square lattice samples. Already without noise the patterns of cracks can become very complex and in particular they can be fractal in the presence of memory with a fractal dimension that depends on the exponent \( \eta \) in the breaking criterion. This phenomenon is due to the competition between
the direction of global stress and the direction of local growth imposed by the lattice anisotropy. Therefore the vectorial nature of the elastic medium leads to crucially different results from what is known to occur in the scalar case of DLA. These differences are likely to subsist when randomness is introduced either quenched in the form of random elastic constants [26] or annealed in the form of breaking probabilities [6, 7].

The influence of quenched disorder in the local breaking thresholds on fracture was analyzed for models which allow for the appearance of many cracks and their mutual influence. For these more realistic models we found numerically scaling laws for the breaking characteristics described by universal exponents. A particularly stable exponent is the one describing how the number of bonds cut scales with system size and which in two dimensions is about 1.7, i.e. close to the value of the fractal dimension of DLA. The distribution of local strain in the fracturing medium is found to become multifractal just before the system breaks apart.

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