Damage Spreading in Spin Glasses.

L. DE ARCANGELOS(*) , A. CONIGLIO(**) and H. J. HERRMANN(*)

(*) SPHT, CEN-Saclay, F-91191 Gif-sur-Yvette, France
(**) Dipartimento di Scienze Fisiche, Università di Napoli
Mostra d'Oltremare Pad. 19, I-80125 Napoli, Italy

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Abstract. – By studying the time evolution of two configurations submitted to the same thermal noise, we investigate phase space properties of the three-dimensional ±J Ising spin glass. Our results for the distance between states are consistent with the picture of phase space having a multivalley structure below Tc. By fixing the damage at one site, we also study the Hamming distance between the two configurations at its critical point, where the distribution of probability of getting damaged n times is found to be multifractal.

Contrary to the mean-field theory [1], which now appears to be quite well established, the investigation of the physical properties of spin glasses in finite dimensions still faces several open problems widely studied both analytically and numerically [2]. Some of the difficulties are due, for instance, to the effects of frustration and disorder which require that numerical simulations must run for very long observation times and be averaged over many configurations of quenched disorder [3]. Nevertheless, the existence of a spin glass phase in three dimensions has been now put into evidence [4], even if its phase space properties are still intensively debated.

The concept of damage or Hamming distance has been used to study the transition to chaos in dynamical systems such as Boolean cellular automata [5]. The idea is to consider two parallel configurations A and B of the same system and evaluate the Hamming distance defined as

\[ D(t) = \frac{1}{4N} \sum_{i} (s_i^A(t) - s_i^B(t))^2, \]

where \( s_i^A(t) \) and \( s_i^B(t) \) are the values of the spins at site i, respectively, in configurations A and B. The distance D(t) depends on its initial value D(0) and it represents how far two configurations evolve in phase space. For two configurations infinitesimally close at \( t = 0 \), D(t) asymptotically will go to a value equal to zero in the so-called frozen phase and to a value different from zero in the chaotic phase.

The same concept was used in a two-dimensional ferromagnetic Ising system at temperature \( T \), where the two configurations were submitted to the same thermal noise [6, 7]. For the heat bath dynamics, for example, at any time step a random number \( z_i \) is drawn for each site i in both configurations and it is compared with the probability \( p_i \) for the spin to be up at that site in the two configurations, with

\[ p_i(t) = \frac{1 + \exp[-2h_i(t)]}{1 + \exp[-2h_i(t)]}, \tag{1} \]
where \( h_i = (1/kT) \sum J_{ij} s_j(t) \), the sum being over the nearest neighbours of site \( i \). For an Ising model evolving with the heat bath dynamics exact results were found [7] relating \( D(t) \) to the magnetization at large \( t \) and the damage at site \( i \) to the spin-spin correlation function.

The study of the damage was also considered for temperature-dependent cellular automata [8] and symmetric [9, 10] and nonsymmetric [10, 11] spin glasses. For the 3d symmetric \( \pm J \) spin glasses \( D(t) \) was evaluated for \( t = 500 \) as a function of the initial distance \( D(0) \) and three phases were observed [9]. For temperature \( T < T_1 \) \( D(t) \) was found to be different from zero but independent of the initial conditions. Finally for \( T > T_1 \) \( D(t) \) is zero.

The value of \( T_1 \) is somewhat larger than the 3d \( \pm J \) spin glass transition temperature \( T_g \sim 1.2 \), whereas \( T_2 \) is smaller than the 3d critical temperature of the pure ferromagnetic Ising model \( T_c \sim 4.51 \). One could be tempted to interpret the first phase as the spin glass phase and the sensitivity to the initial conditions as a consequence of the landscape of valleys within valleys as in mean-field theory.

In mean field [1] each valley in the free energy corresponds to a pure state \( \alpha \) characterized by the set of local magnetizations averaged over the time, \( m_{\alpha i} \) \((i = 1, ..., N)\). The distance between two states \( \alpha \) and \( \beta \) is given by

\[
d_{\alpha \beta} = \frac{1}{4N} \sum_i (m_{\alpha i} - m_{\beta i})^2
\]  

(2)

and the Edwards-Anderson (EA) order parameter associated to each state is given by \( q = q_{\alpha \beta} = 1/N \sum_i m_{\alpha i}^2 \), where the bar stands for an average over the bond configurations.

Whereas the distance \( d_{\alpha \beta} \) depends on \( \alpha \) and \( \beta \) the order parameter is independent of \( \alpha \).

It is not known whether for a 3d Ising spin glass holds the same picture with infinitely many states or a different picture with only two states [12], each obtained from the other by spin inversion. In this second case since there are only two symmetric states, 1 and 2, there is only one distance \( d_{12} \), which due to the symmetry \( m_{1i} = -m_{2i} \) is identical to the EA order parameter, i.e. \( d_{12} = q \). Besides the difficulties connected to the long relaxation times in the spin glass phase, to calculate quantities like (2) there is an intrinsic difficulty in pinning down a particular state \( \alpha \), since there is no apparent symmetry which characterizes the state. This is in contrast to what happens in a pure ferromagnetic Ising model where the two pure states are characterized by an «up» and «down» symmetry and they can therefore be selected by imposing, respectively, the «plus» and «minus» boundary conditions.

To give an indication whether or not the mean-field picture holds for a 3d Ising spin glass, we compare two parallel configurations \( A \) and \( B \) and we evaluate the microscopic Hamming distance

\[
\langle D(t) \rangle = \frac{1}{4N} \sum_i \langle (s_{\alpha i}^A(t) - s_{\alpha i}^B(t))^2 \rangle ,
\]  

(3)

the macroscopic distance between the two configurations

\[
d_{AB}(t) = \frac{1}{4N} \sum_i \langle (s_{\alpha i}^A(t) - s_{\beta i}^A(t))^2 \rangle ,
\]  

(4)

and the EA order parameter in each configuration, \( q_{AA} = \sum_i (s_{\alpha i}^A(t))^2 \) and \( q_{BB} = \sum_i (s_{\beta i}^A(t))^2 \), where the brackets denote the time average over a time interval \([t - \Delta t, t]\) with \( \Delta t = 500 \) time steps and whenever we have a sum over \( i \) we imply also an average over the bonds configurations.
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$$q_{\alpha} = \sum \langle s_\alpha^2 (t) \rangle, \quad \text{with } \Delta t = 500.$$

$$\sum$$ over the bonds

**Model.** - To do so we perform numerical simulations of a 3d Ising spin glass on a cubic lattice of linear size $L$ with periodic boundary conditions. The nearest-neighbours
interactions $J = \pm 1$ are assigned at random, symmetric and quenched in time. The time
evolution of a spin at site $i$ is obtained with the heat bath dynamics in (1) and all the spins
belonging to one of the two sublattices, in which the cubic lattice separates, are updated in
parallel.

Let us start from an initial configuration of spins, obtained after a given thermalization
time of typically some thousand steps, and quenched interactions in system $A$. We construct
then its parallel image, system $B$, with the same interactions $J$'s and all the spins in the same
state as $A$ but a finite fraction which is over turned and constitutes the initial damage $D(0).

The two systems $A$ and $B$ evolve in time according to the dynamics (1) with the same random
number determining the state of spin $i$ in configurations $A$ and $B$. We average then over the
quenched disorder by considering several samples of interactions and initial configurations
for the various system sizes considered. System sizes from $L = 6$ up to $L = 22$ are analysed
and the average is performed from over 700 samples for $L = 6$ to 20 samples for $L = 22.$

**Results.** - Let us first look at the EA order parameter. We find that (fig. 1) $q_{AA} = q_{BB} = q$,
whereas $d_{AB}$ depends on the initial damage $D(0)$. One can identify the spin glass temperature
$T_g$ as the temperature at which the order parameter $q$ and $d_{AB}$ go to zero. This temperature
coincides roughly with the temperature $T_1$, where the damage becomes independent of the
initial conditions. Our data for $L = 10$ and $t = 2000$ are consistent with $T_g \sim 1.9$ and $T_1 \sim 1.8$,
to be compared with the most recent numerical result $T_g \sim 1.2$. This shows that for $t = 2000$
we are still very far from equilibrium. The same analysis performed in an external magnetic
field $h = 1$ gives $T_g \sim 2.1$ and $T_1 \sim 4.3$ within ten percent.

We have checked how stable the phase diagram of fig. 1 is. In fig. 2 we show for the
system size $L = 30$ the value of $q_{AA}, q_{BB}, d_{AB}, \langle D(t) \rangle$ as a function of time for $T = 0.9$ and an
initial damage $D(0) = 1/2$. We note that $q_{AA}$ and $q_{BB}$ are steadily constant, whereas $d_{AB}$ and
$\langle D(t) \rangle$ slowly decrease in time. This effect may be due to the fact that for extremely long
times and for any finite system size the configuration will evolve toward the same
microscopic state and therefore both $d_{AB}$ and $\langle D(t) \rangle$ will eventually go to zero, or to the fact
that extremely long times are needed before the configuration settles into a valley and that
some configurations can relax faster than other ones.

One can change the relaxation behaviour by fixing the damage on the boundary. We have
checked for various temperatures that, by fixing two lines random and opposite on the
boundaries of configuration $A$ and $B$, both $\langle D(t) \rangle$ and $d_{AB}$ asymptotically stay constant
within a time of 40 000 steps (fig. 2b).

The finding that $d_{AB}$ depends on $D(0)$ and is different from $q_{AA}$ and $q_{BB}$ seems consistent
with the scenario present in the mean-field theory of infinitely many states. In fact the two
parallel configurations, being prepared at $t = 0$ in two different microscopic states, may
evolve into two different macroscopic states. Now under the hypothesis that only two states
are present, the distance $d_{AB}$ between the two macroscopic states would either be zero
(corresponding to the case in which $A$ and $B$ are in the same state) or equal to the EA order
parameter $q_{AA}$ (corresponding to the case in which $A$ and $B$ are in opposite states). Figure 2
seems to exclude this picture and support the hypothesis of infinitely many states as
indicated also in fig. 1, where the average distance between the states in $A$ and $B$ depends
on the initial damage $D(0)$.

We wish to stress again, however, that full thermalization has not been achieved and that
we have not analysed the dependence of the various quantities on the time interval $\Delta t$.
Therefore the above interpretation must be considered as a working scheme which needs a
more careful analysis [14] to be fully tested.
Fig. 1. — a) The EA order parameters $q_{AA}$ and $q_{BB}$ as a function of the temperature $T$ for 200 configurations of the system size $L = 10$ after 2000 time steps for differential initial damages. b) The microscopic Hamming distance $\langle D(0) \rangle$ and the macroscopic distance $d_{AB}$ as a function of the temperature $T$ for the same system size, statistics, time steps and for different initial damages $D(0)$.

Fig. 2. — The EA order parameters $q_{AA}$ and $q_{BB}$, the Hamming distance $\langle D(0) \rangle$ and the distance $d_{AB}$ as a function of time for one configuration of the system size $L = 30$, a) for an initial damage $D(0) = 1/2$ at $T = 0.8$, and b) for two lines fixed random and opposite in configurations A and B at $T = 1.7$. 
Multifractality. - We now address the question whether multifractality [15] is present at $T_c$. We fix a site at the origin $s_0^z = +1$ in configuration $A$ and we monitor for each site $i$ the number of times $n_i$ the site $i$ changes from a healed configuration to a damaged one, and we evaluate $p_i = n_i / \sum n_i$, the probability for each given site $i$ to be damaged from a healed state. The probability $p_i$ is a measure of the frequency at which the site $i$ gets damaged [16].

Taking the moments

$$M(q) = \sum_i p_i^q \sim L^{-\tau(q)}$$

for different system sizes at the temperature $T_c = 4.51$ we find (fig. 3) that $\tau(q)$ changes non-linearly with $q$ implying a multifractal structure. Physically this means that among the sites which are damaged in the course of time there are fractal subsets which are damaged more frequently than others. For $q = 0$ we find that $\tau(0) = \alpha = 3$ implying that a finite fraction of sites is damaged at least once.

We have also found multifractality using the different measure $\tilde{p}_i = \tilde{n}_i / \sum \tilde{n}_i$, where $\tilde{n}_i$ is the number of times the damage changes from a $(+ -)$ configuration to a $(- +)$ one. The probability $\tilde{p}_i$ also is a measure of the frequency of changing damage. The two frequencies are somehow related and the moments of $\tilde{p}_i$ behave similarly to the moments of $p_i$, the latter

Fig. 3. - Log-log plot of the quantities $[M(q)/M(0)]^{1/q} \times L$ for the 3d Ising spin glass at $T = T_c = 4.51$. The different $q$ values are $q = 0$ ($\circ$), 1 ($\bullet$), 2 ($\nu$), 3 ($\triangle$), 4 ($\triangledown$). The spreading out of the straight lines indicates that the critical exponents $\tau(q)$ do not have a linear dependence on $q$.

Fig. 4. - Log-log plot of the quantities $[M(q)/M(0)]^{1/q} \times L$ for the 3d pure Ising ferromagnet at $T = T_c = 4.51$. The symbols are as in fig. 3.
have however a better statistics since they are constructed on the basis of more numerous events.

The reason for studying the distribution of $p_i$ and their moments is due to the fact that for a pure ferromagnet or antiferromagnet the probability for a site to change damage is zero. Therefore, for high temperature in the paramagnetic phase one expects $d(p)$ to be sharply peaked near $p = 0$. As the temperature is lowered, frustration sets in resulting in a more frequent change in the correlation signs and the distribution can become quite wide giving rise to multifractality. We have also measured the moments of the probability at the spin glass transition and we found there no multiplicity of critical exponents but the moments follow constant gap scaling.

The appearance of multifractality at $T_c$ is an even more striking result, since it is a property found only for three-dimensional spin glasses. In fact, we have performed numerical simulations also for the three-dimensional pure Ising ferromagnet at $T_c$ and we have found that the moments in eq. (5) follow constant gap scaling (fig. 4). Moreover, the same analysis performed for the two-dimensional $\pm J$ spin glass, both at the pure-ferromagnet critical temperature $T_c \sim 2.21$ and at the onset of the chaotic phase $T_c \sim 1.5 + 1.7$, also show no multifractal behaviour. Preliminary results [13] have also found multifractality in the 3d Ising spin glass in a magnetic field $h = 1$ at the onset of the chaotic phase $T_c \sim 4.3$.

Conclusions. — We have investigated the phase space properties of the three-dimensional Ising spin glass by looking at the distance of two parallel configurations submitted to the same noise. We have found that the macroscopic distance between states $d_{\text{AN}}$ goes to zero at a temperature close to $T_{\text{AN}}$ and, as the Hamming distance, it is sensitive to the initial conditions. We have also studied the fractal properties of the distribution of probability of being damaged $n$ times and we have found that it is multifractal at $T_c$.

The approach presented in this paper is suitable to give more insights in the study of phase space properties of spin glasses. However, a more careful numerical analysis needs to be done to fully test the presented ideas.

REFERENCES