1. INTRODUCTION

1.1. Some words on the history.

Around 1950 J. von Neumann introduced cellular automata in order to mimic a machine that should be self-replicating and a universal computer (the second is also called Turing machine; see references in first article of Reference 3). He found an automation that had this property but it was so complex (29 states per cell, 200,000 cells) that even today nobody has implemented it on a computer. Subsequently followers have found some simpler versions and as a by-product many other automata were invented ranging from those that are just pattern replicating (R. Friedkin) to those that are just Turing machines like J. H. Conway's famous game of life. A nice account on these events is given in Reference 1.

Cellular automata were introduced into physics by S. Wolfram who put up the thesis that all laws of nature could be given in terms of cellular automata. He gave the first systematic classification and proposed measuring physical properties like entropy, Lyapunov exponents and transients. Another pioneering effort was made by the Information Mechanics Group at M.I.T. who built and commercialized the first dedicated cellular automata computers.

Around 1983 a considerable amount of statistical physicists started to work on the field studying carefully some automata for which they were able to find physical realizations. Some of these automata could be mapped on to known models from statistical mechanics and other automata were found to exhibit novel types of phase transitions. It is mainly this last part on which I want to put the emphasis in this article.

1.2 Definitions

Cellular automata are dynamical models which as opposed to the usual...
Hamiltonian is given and therefore cellular automata do not necessarily evolve towards an equilibrium. We can expect a rich behavior. Cellular automata are such a general concept that one can compare them e.g. to the area of mathematics of differential equations.

For a given \( r \) and \( k \) a total of \( r^k \) rules are possible. This is so because for \( k \) inputs there are a total of \( r^k \) possible input configurations and a rule is uniquely defined by giving to each of these configurations one of the \( r \) possible output values. The rules can then be identified according to Wolfram by a number

\[
C = \sum_{n=0}^{k-1} r^n f(n)
\]

(3)

The easiest examples are one-dimensional automata where the input come just from the same cell and its two nearest-neighbors (\( k = 3 \)) and one has binary variables (\( r = 2 \)). For this case one can find very useful tables in the appendix of Reference 3 which summarize their properties. Specifically let us take for this case the rules with \( C = 90 \) and \( C = 22 \) which decomposed in binary numbers according to the possible neighborhood configurations are given by

<table>
<thead>
<tr>
<th>rule</th>
<th>111</th>
<th>110</th>
<th>101</th>
<th>100</th>
<th>011</th>
<th>010</th>
<th>001</th>
<th>000</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**TABLE I**

We see that rule 90 gives "1" exactly when the two neighboring cells are in different states and rule 22 gives "1" exactly when only one of the three inputs is "1". All binary rules can also be expressed as Boolean functions like

\[
\text{rule } 90 = (\sigma_4 \land \sigma_3) \lor (\sigma_2 \land \sigma_1) = \sigma_4 \lor \sigma_3
\]

\[
\text{rule } 22 = (\sigma_4 \land \sigma_3 \land \sigma_2) \lor (\sigma_1 \land \sigma_0 \land \sigma_5) \lor (\bar{\sigma}_1 \land \bar{\sigma}_0 \land \sigma_2)
\]

(4)

where \( \lor, \land, \oplus \) and \( \bar{\cdot} \) mean the logical operations or, and, exclusive or, and complement defined by

dynamical systems theory (Rögen and so on) have many but discrete degrees of freedom. Actually there is a threefold discreteness in cellular automata: discrete space, discrete time and discrete number of states. Each discrete element of space (also called cell) has a given number \( k \) of inputs and a rule which defines its state at the next time step as a function of the inputs. The rules are simultaneously updated. To define the inputs one can make a diagram like the one shown in Figure 1. Usually, however, one will look at regular lattices where the inputs come just from itself, nearest and sometimes also next-nearest neighbors. Let us denote by \( \sigma_i = 0, \ldots, r - 1 \) the variable on cell \( i \) which can have \( r \) different states. (Note that my definition of \( r \) and \( k \) is just the opposite of Wolfram's but agrees with Kaufman's definition of \( k \).) A deterministic cellular automaton is then simply defined for each cell \( i \) through

\[
\sigma_i(t+1) = f_j(\sigma_j(t)) \quad j = 1, \ldots, k
\]

(1)

where the functions \( f_j \) will have values \( 0, \ldots, r - 1 \). Some times one wants to have also a dependence on former times and in the case of probabilistic automata there will also be a dependence on a random number \( z \) so that a more general definition is given by

\[
\sigma_i(t+1) = f_j(\sigma_j(t), \ldots, \sigma_j(t-\tau)) \quad j = 1, \ldots, k \quad z
\]

(2)

where \( \tau \) is a finite integer.

**FIGURE 1**

Diagram of input for \( k = 3 \) and a system of four cells.

From the point of view of a statistical physicist we recognize e.g. Küber dynamics as a special case of Equation 2. But in this more general context no
\[ \begin{array}{c|c|c|c|c}
\hline
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
\hline
\end{array} \quad \begin{array}{c|c|c|c|c}
\hline
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
\end{array} \]

Note that in Equation 4.0-1 (\(o_2\)) is the left (right) nearest-neighbor of \(o_3\).

It is very important for the understanding of a specific automaton to know which are the symmetries it can have. So, for instance, inventing all zeroes to ones and all ones to zeroes in the configurations and outputs of Table 1, rule 90 becomes rule 126 and rule 22 becomes rule 151, while e.g. rule 51 remains invariant under this spin-flip symmetry. Another symmetry operation one can perform is to interchange left and right cells ("reflection"). Under this symmetry, rules 22 and 90 happen to be both invariant. It is usually enough to take only one representative rule with respect to both of the above-mentioned symmetries, the "minimal representative", because the behavior of the rule that one obtains after performing the symmetry operation can then be obtained very simply. There are 80 minimal representatives out of the 256 rules for \(k = 3\) and \(r = 2\). If one only takes, however, the rules that are invariant under the two symmetry operations one gets what Wolfram calls the "legal" rules and there are only 32 of them for \(k = 3\) and \(r = 2\).

Other subgroups of cellular automata defined according to their symmetries have been called "additive", "totalistic", "outer totalistic", "forcing", "reversible", etc. Let me only say that a rule is "totalistic" if it depends only on the sum of the input values. Out of the \(10^{14}\) rules for \(k = 9\) and \(r = 2\) only \(10^4\) are legal and only \(512\) totalistic.

1.3. General behavior of cellular automata

If one observes the space-time patterns formed by different automata starting from a random initial configuration, one sees striking differences as shown in Fig. 2 where some one-dimensional rules for \(k = 5\), \(r = 2\) are displayed. Wolfram proposed a classification of the behavior into four classes:

- Class 1 (rules 4 and 16 in Figure 2) has a homogeneous final state.
- Class 2 (rules 8 and 24 in Figure 2) yields spatially separated periodic structures.
- Class 3 (rules 2, 6, 10, 12, 14, 18 and 22) gives disordered patterns that have holes of many sizes.
- Class 4 (rule 20 in Figure 2) gives complex localized structures.

In the language of dynamical systems, Wolfram associates class 1 to fixed points, class 2 to limit cycles, class 3 to chaos with strange attractor and class 4 to a new class that does not exist in dynamical systems with only few degrees of freedom. The definition of these classes at this stage is quite vague and it seems very difficult to assert in general, i.e. for all initial configurations and arbitrarily big spatial sizes, to which class an automaton belongs. It has actually been proven after mathematically sharpening the definition of the classes that it is undecidable whether for a given automaton all configurations will go to a homogeneous state. It is therefore undecidable to which class an automaton belongs. Undecidable means that one can only obtain the information by actually performing the time evolution.

**FIGURE 2**

For the rules 2 to 24 with \(k = 5\), \(r = 2\) one sees a site in black if it is one and in white if it is zero. The one-dimensional space is given horizontally and subsequent times are shown on subsequent lines. This figure was taken from Reference 3.
The automata of class 4 are particularly intriguing. In Figure 3 we see some patterns obtained from the rule 20 of Figure 2 using different initial conditions. For chains of 20 sites in more than 90% of the cases the final state is homogeneous. But in some cases persistent structures are formed. For this automaton 11 different of these structures have been identified up to now, some appearing with a frequency smaller than $10^{-5}$. Some of them are not fixed in space but move with constant velocity and can be used to transmit information. It is these kind of structures (together perhaps with a “glider gun") that are needed to construct a Turing machine and for this reason Wolfram conjectures that the class 4 automata might be universal computers. The game of Life which also belongs to class 4 can be formulated in three dimensions and there it can produce more than 100 different kinds of persistent structures that can be contained within a $4^3$ cube.

![Patterns](image)

**Figure 3**

Patterns from the rule 20 of Figure 2 ($k = 5$, $r = 2$) for different initial configurations. Figure taken from Reference 3.

To characterize the patterns generated by cellular automata several types of entropies were proposed by Wolfram. One looks at sequences of the states of one cell in $T$ consecutive time-steps or of a string of $x$ spatially adjacent cells at a given time and calls $p_j$ the probability to find the sequence $j$ as a temporal string of length $T$ and $p_j^x$ the probability to find sequence $j$ as a spatial string of length $x$.

Then measure entropies are defined as

$$S_{T}^{(x)} = -\frac{1}{x} \sum_j p_j^x \log_x p_j^x$$

$$S_{x}^{(T)} = -\frac{1}{T} \sum_j p_j^x \log_T p_j^x$$

where the average $\langle \ldots \rangle$ is taken over all initial configurations. One can obtain entropies independent on the length of the sequence as

$$S_x = \lim_{x \to \infty} S_{T}^{(x)} \quad \text{and} \quad S_T = \lim_{T \to \infty} S_{x}^{(T)}$$

Wolfram's classification can then mean

Class 1 $\equiv S_x = S_T = 0$

Class 2 $\equiv S_x \neq 0$, $S_T = 0$

Class 3 or Class 4 $\equiv S_x \neq 0$, $S_T \neq 0$

as can be guessed looking at Figure 2.

The numerical determination of the entropies is unfortunately very tedious and the convergence in Equation 7 is often very slow. For this reason they are not used very often.

The relation between cellular automata and dynamical systems of few degrees of freedom can be investigated by looking at quantities that are accessible in both cases like the Lyapunov exponent $\lambda$. A good definition of $\lambda$ in a cellular automaton is to take two initial configurations that differ just in the state of one cell and to plot the cells on which the state differs in subsequent times in black as shown in Figure 4 for rule 22 in one-dimension, $k = 3$, $r = 2$. Then $\lambda$ is just the inverse of the slope of the envelope of the resulting...
triangle (Figure 4). Physically $\lambda$ is the speed with which the damage between the two configurations propagates. Grassberger found for rule 22 the value $\lambda = 0.7660 \pm 0.0003$ (see Reference 7) which is clearly positive and the automaton therefore chaotic. On the other hand he measured $S_{e}(X) \sim t^{-0.08}$ and $S_{e}(T) \sim t^{-0.18}$, which means that the entropies slowly decay to zero and after a sufficiently large $T$ one will always have $S_{e}(T) < S_{e}(2^{T})$. For dynamical systems with few degrees of freedom one expects an equality $S_{e}(T) = S_{e}(2^{T})$? So there can be cellular automata for which there is chaos according to the Lyapunov exponent but not according to the entropies.

**Difference pattern between two patterns that at time $t = 0$ agree everywhere except on a single point and are random otherwise. Taken from Reference 7.**

![Difference pattern between two patterns that at time $t = 0$ agree everywhere except on a single point and are random otherwise.](image)

Grassberger's article in Reference 1 points out another curiosity of cellular automata: One can transform a cellular automaton of binary variables given by \ldots G_2 \ 0 \ 0 \ G_2 \ 0 \ \ldots into a dynamical system of two real variables $x$ and $y$ via $X = 0.5 \ 0 \ G_3$ and $0.5 \ G_3$.

Then, various rules define strange attractors. But from time to time their time-evolution falls outside the attractor with a frequency that decays in time like $1/\sqrt{t}$. Such a behavior is also new compared to standard dynamical systems.

**Cellular automata**

We can summarize our comparison of cellular automata with the usual dynamical systems of few degrees of freedom in that cellular automata show a similar but richer behavior: 1) They can belong to Class 4; 2) Entropies can vanish having a positive Lyapunov exponent; 3) Strange attractors can be imperfect.

1.4. Transients

Before falling into a fixed point or another attractor, cellular automata usually go through a transient. If one considers the attractor to represent some equilibrium, one can see these transients like in statistical mechanics as a relaxation towards equilibrium.

For rule 22, $k = 3$, $r = 2$, the "magnetization", i.e. the average number of '1's in equilibrium, has been found to be $m(\infty) = 0.35096 \pm 0.00004$ (see Reference 7). So Zabolitsky investigated how starting with a random configuration ($m(0) = 0.5$) equilibrium is reached. In Figure 5 we see for rule 22 how the magnetization $m(t)$ approaches $m(\infty)$ averaged over 350 samples. The statistical error is of the order of $5 \times 10^{-3}$ as can be seen from the flat part at right. From the left side of the figure one sees on one hand that the data on the average follow a straight line, i.e. an exponential decay law

$$|m(t) - m(\infty)| \approx e^{-t/T} \quad (6)$$

with a relaxation time $T = 7.7$. On the other hand, the data show an extremely complex structure which is not due to statistical fluctuations but which is an effect of the deterministic nature of the automaton. This is a quite generic feature of deterministic cellular automata: they can be described very well by laws that we know from statistical mechanics like Equation 8 but only if one disregards the complicated fine-structure.

Similar situations have been observed for the same rule 22 for the spatial correlation function $C_3 = \langle G_3 \ G_5 \ G_7 \rangle > \langle G_3 \rangle \langle G_5 \rangle \langle G_7 \rangle$; and for the size dependence of $T$. If $p$ is the fraction of '1's in the initial configuration one can even find critical slowing down, i.e. $T \to \infty$, for $p \to 0$ for this rule.

For small sizes a systematic study for many automata is given in the appendix of Reference 3 and one finds very strong irregularities in the transients as well as in the cycle-lengths.

As a final example on transients let me mention the case of rule 184 for $k = 3$, $r = 2$ in one dimension. If one interprets a '1' as a positive slope and a '0' as a negative slope of absolute value unity, this automaton describes a profile which at each time-step is covered by one layer (see schematic picture). So if one starts with a random initial configuration, the average
2. CELLULAR AUTOMATA COMPUTERS

For a computer designer cellular automata are ideal because they are simple in two respects. Since the rules have fixed inputs and are updated simultaneously they allow for a parallel architecture. Since they only have a small number of discrete states one can use just logical functions and does not need arithmetic operations, not to talk of floating point operations. For this reason several cellular automata have been built and are under construction.

Most famous has been Toffoli's CAM 6 (see Reference 10), which is quite inexpensive and user friendly. It is card which is inserted into the back of an IBM compatible PC and commanded by a soft resident on the disk of the PC. It updates a 256 system of four planes of binary variables 60 times per second, such that it is ideally suited for color display. A similar machine was also built in Florence.

In only 11 months the RAP 1 machine was completed at E.N.S. in Paris11. Comparable in speed to the CAM 6 it is particularly tailored to lattice gas calculations on the triangular lattice. Its strength is also displaying on a screen. All these machines internally use sequential updating and not parallel updating. The weakness of these machines is when it comes to actually calculate quantities like correlation functions, Lyapunov exponents, etc., because the number crunching must be done on the host, namely the PC. These kind of calculations then form the natural bottleneck.

The next generation of cellular automata computers, with parallel updating and much larger systems, is coming. A first promising example is the Connection Machine (see e.g. article by W. D. Hills in Reference 1) which can configure any network due to its switching network. A CAM 7, 17,000 times bigger and faster than the CAM 6, is being designed.

3. MODELING THERMODYNAMICS WITH CELLULAR AUTOMATA

3.1 Reversible automata

If one wants to model the microscopic behavior of nature one has to use time-reversible automata since the elementary laws of nature are time reversible. An automaton $f_1(g_1(t))$ is called time-reversible if there exists an automaton $g_1$ such that $g_1(f_1(t)) = f_1(g_1(t))$. A necessary condition for an automaton to be time-reversible is that each configuration has a unique predecessor which implies that the automaton cannot have transients. Practically all automata we have seen up to now have transients and are therefore not time-reversible.

One way to construct time-reversible automata is the following12: put two
binary variables \( \sigma_i \) and \( \sigma_j \) into each cell so that effectively one has an automaton of \( r = 4 \). Then

\[
R : \begin{cases}
\sigma_i(t + 1) = f_i(\sigma_i(t)) \oplus \sigma_i(t) \\
\sigma_i(t + 1) = \sigma_i(t)
\end{cases}
\]

(10)

is a time-reversible automaton where \( f_i \) is of the same kind as in Equation 1.

This can be seen by decomposing \( R \) like \( R = BA \) with

\[
A : \begin{cases}
\sigma_i(t + \frac{1}{2}) = f_i(\sigma_i(t)) \oplus \sigma_i(t) \\
\sigma_i(t + \frac{1}{2}) = \sigma_i(t)
\end{cases}
\]

(11a)

and

\[
B : \begin{cases}
\sigma_i(t + \frac{1}{2}) = \sigma_i(t) \\
\sigma_i(t + \frac{1}{2}) = \sigma_i(t)
\end{cases}
\]

(11b)

one sees immediately that

\[
AA = 1 \quad \text{and} \quad BB = 1
\]

(1)

is the identity also called "tautology")

If one now defines \( \tilde{R} = BA \) then \( \tilde{R} \) is the time-reversed automaton to \( R \) since

\[
\tilde{R} \tilde{R} \tilde{R} \tilde{R} \tilde{R} = BABABAABA = 1
\]

Indeed one sees that at the time one wants to reverse the time one just has to make one operation \( B \) more and then one can apply the same automaton \( R \):

\[
\tilde{R} \tilde{R} \tilde{R} \tilde{R} \tilde{R} = BAABABABA = 1
\]

The drawback of the automaton of Equation 10 is that one has two binary variables in each cell; it would be preferable to have a simple cell. It is possible to split each cell in two cells having spatially a checkerboard structure where the \( \sigma_i \)'s are on cells of one sublattice and the \( \sigma_i \)'s on cells of the other sublattice as illustrated in Figure 6. To do this the lattice must have even coordination number (i.e., even number of nearest-neighbors and an even number of sites in each direction)

\[
\begin{array}{cccc}
\sigma_{i-1} & \sigma_{i} & \hat{\sigma}_i & \sigma_{i+1} \\
\end{array}
\]

(a)

\[\begin{array}{cccc}
\sigma_{i-1} & \sigma_{i} & \hat{\sigma}_i & \sigma_{i+1} \\
\end{array}\]

(3)

Sublattice structure in (a) one dimension and (b) on a square lattice

If now one defines in one dimension

\[
B_1 : \begin{cases}
\sigma_i(t + \frac{1}{2}) = \hat{\sigma}_i(t) \\
\sigma_i(t + \frac{1}{2}) = \sigma_{i+1}(t)
\end{cases}
\]

and

\[
B_2 : \begin{cases}
\sigma_i(t + \frac{1}{2}) = \hat{\sigma}_i(t) \\
\sigma_i(t + \frac{1}{2}) = \sigma_i(t)
\end{cases}
\]

(12)

then the automaton \( R = B_1 B \) has as time reversed automaton \( \tilde{R} = A B_2 \) similarly one makes on the square lattice a shift in both coordinates (in \( B_1 : \sigma_{i,j} = \sigma_{i,j+1} \), etc.). If one has periodic boundary conditions the shift in Equation 12 makes no effect. We see that a sublattice structure is essential if one wants to have time-reversibility.

3.2 Conserved quantities

An invariance very often produces conservation laws. So it makes sense to look for additive conserved quantities in the above defined family of time-reversible automata. For \( k = 3, r = 2 \) this was done in one dimension in Reference 12 and among the 88 minimal representatives 41 had no conserved quantity, 7 had only a global conserved quantity and 40 had even a local conserved quantity. Below, an example for a local and global conserved quantity will be given.

Here we want to focus on a special choice of automata which were first introduced by Vichniac in Reference 1, namely having in Equation 11a the totalistic rule:

\[
f(x) = 1 \iff x = q \quad \text{with} \quad x = \sum_{j=1}^{k} \sigma_j(t)
\]

(13)

where \( 0 \leq q \leq k \) is a parameter. Feneau\(^13\) showed for this automaton defined on sublattices like in Figure 6 the quantity
\[ E(t) = \sum_{i,j} \sigma_i(t) \delta_i(t) - \sum_i (\sigma_i(t)) + \delta_i(t)q \]  (14)

(Where \(<i,j>\) means pairs of nearest-neighbors) is conserved. To see this let us note that one can rewrite the rule of Equation 10 using

\[ \sigma_i(t+1) = f(\delta_j(t)) \oplus \delta_i(t) = \delta_i(t)+(1-2\delta_i(t))f(\delta_j(t)) \]  (15)

which can be checked by inspection.

Next one can write rearranging the terms in Equation 14:

\[ E(t+1) = \sum_i \sigma_i(t+1) \left[ \sum_j \delta_j(t+1) - q \right] - \sum_i \delta_i(t+1)q \]  (16)

Inserting (15) and \(\delta_i(t+1) = \sigma_i(t)\) into Equation 16 and using that \(f(\delta_j(t))\) is always zero when the square brackets in Equation (16) are non-zero one finds \(E(t+1) = E(t)\) as to be shown.

In the same way one can also show that

\[ R_i(t) = \sum_j \delta_j(t) \sigma_j(t) - \sum_j \delta_j(t)q - \delta_i(t)q \]  (17)

where the sums over \(j\) only sum over the nearest-neighbors of \(i\), is also conserved. So one has in Equation 17 even a locally conserved quantity.

In one dimension with periodic boundary conditions and \(q = 1\) one can interpret each bond for which the two neighboring cells are in different states as particles (see Figure 7). When the rule is applied, the particles move one clockwise and some counterclockwise depending on their original position on the lattice. The conservation laws of Equation 17 mean that particles conserve their velocity. For this reason the particles do not see each other and one has a one-dimensional ideal gas. This is the simplest thermodynamical model we can imagine. Since the phase space is broken up into cycles which cannot be longer than the length of the chain (because then all possible configurations would occur twice) the dynamics is not ergodic in the usual sense. Still we imagine it to reasonably well reproduce thermodynamic behavior.

**Cellular automata**

Closed chain of 5 sites, one sublattice marked by boxes. The two particles are given as crosses.

### 3.3 QSR

Automaton (13) with \(q = 2\) on the square lattice has been called QSR (Q = quadrato = number of neighbors, 2 = q, R = reversible), and it has got some attention due to its relation to the Ising model. The invariant of Equation 14 can be rewritten as

\[ -H/\tau = \sum_{<i,j>} \sigma_i \delta_j + h \sum_i (\sigma_i + \delta_i) + \text{const.} \]  (18)

where \(h = 2x - 4q\) is called the field and the variables \(\sigma_i = 1 - 2\delta_i\) have values \(+1\). This is just the Hamiltonian of the Ising model. So QSR does just conserve the energy of the Ising model with no field. Therefore one can choose initial configurations with a given value of the Ising energy and measure the magnetization, i.e. the fraction of \(1\)'s in QSR. In this way one gets a relation between energy and magnetization in QSR and discovers the striking fact that it agrees within numerical accuracy with the exactly known relation for the Ising model on a square lattice as can be seen in Figure 8.

This empirical finding has been confirmed also in three dimensions but no proof or deeper understanding is yet available. Still one might consider QSR as a candidate for microscopic dynamics of the Ising model.

The big advantage of using QSR is computational. By using multi-spin coding and updating in parallel 64 cells using a bit by bit logical operation of the form

\[ \sigma_i(t+1) = (\sigma_i(t) \oplus (\sigma_1 \oplus \sigma_2 \oplus \sigma_3) \oplus (\sigma_4 \oplus \sigma_5 \oplus \sigma_6)) \]  (19)
where $\sigma_1, \ldots, \sigma_4$ are the neighbors of $\sigma_i$ at time $t$, one can achieve updates of 4200 million updates per second in systems of $1.5 \times 10^{10}$ sites compared to the world record of 120 million updates per second for sizes $3 \times 10^8$ obtained with Metropolis Monte Carlo.

![Diagram](image)

**Figure 8**
Relation between magnetization and energy for QSR. The full line is the exact relation for the Ising model. Different sizes have different symbols. Taken from Reference 14.

Due to the additional conservation law in QSR compared to a canonical dynamics the relaxation to equilibrium is slower. This can be seen in Figure 9. Since the system size used for QSR is much larger, one sees nearly no statistical fluctuations which is a big advantage of QSR. The relaxation of QSR is about 15 times slower than for Metropolis in the case of Figure 9; so, one only gains a factor of 2.5 instead of 50 in computer time.

![Diagram](image)

**Figure 9**
Relaxation towards equilibrium of the magnetization $M$ at temperature $T = 1.09235 \ T_c$. Right curve QSR for a system 123000$^2$ and left curve Metropolis with a size 8300$^2$. Taken from Reference 16.

3.4 Heat transport in QSR

The energy conservation in QSR allows for a measurement of energy flow$^{17}$. One considers a finite system and has periodic horizontal boundary conditions and on top and bottom one imposes energies $E$ and $E + \Delta E$ respectively (see schematic picture below) by fixing a density of bonds that always cost energy. Letting QSR evolve one finds for $E > E_c = -\sqrt{2}$, i.e., in the paramagnetic phase, an energy flow of $Q$ and one defines the heat transport coefficient $K$ by

$$K = Q(\Delta E \cdot L \cdot \Delta t)^{-1} \quad (28)$$

where $L$ is the horizontal length of the system and $\Delta t$ is the time interval over which the flow was measured.

![Diagram](image)
One finds \( \text{that there are two contributions to } K; \) one which goes like \( E - E_p \) to zero when the critical point is approached and which is due to diffusing of energy-carrying bonds and a second contribution which is due to "highways". An example of such a highways is shown in Figure 10. It is essentially a grain-boundary which delivers one energy-unit from one side to the other at each second time-step. This efficient way of energy transport has a maximum of occurrence well inside the paramagnetic phase and disappears in the thermodynamic limit.

![Figure 10](image)

A "highway" at five consecutive time-steps.

3.5 Period-transition and ergodicity in QSR

Since QSR is a deterministic automaton it moves on periodic cycles in phase space. One way to analyze these periods is to look at the period of each single site. In Figure 11 we see a configuration at small energies which shows that the sites of a given period are lumped together in clusters and separated by a large amount of sites that do not move at all (period = 1). These dynamical clusters show which sites are correlated in time to one another but it is not easy to define them properly unless a detailed study of the period of each cluster is made. We are not going to do that. Easy to measure is the average period \(< t >\) of a single site averaged over all the local periods less than \( \infty \).

In Figure 12 we show how \(< t >\) diverges for increasing \( t \) at an energy around \( E_p = -1.75 \). This indicates a critical point: at energies below \( E_p \) one finds only finite periods and above \( E_p \) sites with arbitrary large periods (in the thermodynamic limit). This transition very slightly moves as seen from the right side of Figure 12 and it is not sure whether \( E_p \) is actually a little smaller. Since the clusters are confined to rectangles and since they do not look fractal some analogies exist to bootstrap percolation. This is a percolation problem in which each site having more than one neighbor will be occupied.
the deterministic dynamics introduced by M. Crentz's\textsuperscript{20} which is a generalization of QSR. In Crentz's automaton to each site an additional heat reservoir is attached mimicking a kinetic energy and which consists of additional bits, i.e. a variable of $2^m$ possible values, on which the initial energy is set. A site is flipped if its heat reservoir can absorb or furnish the energy difference between the old and the new local configuration. So evidently many more flips are possible and the strict local energy conservation of QSR is now relaxed to the conservation of the sum of kinetic and potential energy. QSR corresponds to the case $m = 0$. Still this dynamics is not ergodic in the strict sense. The question of ergodicity in deterministic automata that are supposed to solve thermodynamics is not solved. Perhaps a weaker definition of ergodicity would be appropriate.

4. SELF-ORGANIZED CRITICALITY

In the last year the notion of self-organized criticality has been propagated by P. Bak and collaborators\textsuperscript{21}. It says that many systems in nature submitted to some external excitation naturally evolve into a steady state which is critical in the sense that it displays spatial and temporal long range correlations like fractals or $1/f$ noise. As an example one may visualize a sandpile on which constantly new sand is dropped. It evolves towards a critical slope on which avalanches of all sizes can be produced by a single excitation.

Threshold cellular automata have been proposed to model this situation\textsuperscript{22} which on each site $i$ have an integer variable $Z_i \geq 0$ and there is a threshold $Z_c$ which should be at least $2d - 1$, where $d$ is the dimension of space. Let us first discuss the model in one dimension: If a site has $Z_i > Z_c$ then $Z_i \rightarrow Z_i - 2$ and $Z_i \rightarrow Z_i + 1$ so that the automaton rule is:

$$Z_i(t+1) = Z_i(t) - 2 \theta(Z_i(t) - Z_c) + \sum_{nn} \theta(Z_{nn}(t) - Z_c)$$

(21)

where $\theta(x)$, the Heaviside function, is zero for $x \leq 0$ and one for $x > 0$. So the sum over the $Z_i$ is conserved. In the picture of the sandpile the $Z_i$ is supposed to be the local slope. To make this model meaningful the boundary conditions are very important. Two types of situations at the boundary be considered:

I. same rule for all sites; $Z_{1,0} = 0$

(22)

II. $Z_0 \rightarrow Z_0 - 1$ and $Z_{d-1} \rightarrow Z_{d-1} + 1$ for $Z_d > Z_c$
Condition I does not conserve the sum of the \( Z_i \) at the boundary and corresponds to a closed end in the sandpile analogy. Condition II conserves the sum of \( Z_i \) and corresponds to an open end of the sandpile. Also several types of excitation are possible:

a) \( Z_i \rightarrow Z_i + 1 \)  
\[ (23) \]

b) \( Z_i \rightarrow Z_i + 1 \) and \( Z_{i+1} \rightarrow Z_{i+1} - 1 \)

The first corresponds to locally increasing the slope of the sandpile which involves a major rearrangement of the sand while the second one corresponds just to add a grain of sand and is therefore much easier to implement.

It is easy to understand that if particles are constantly added (excitation of type b) the system will settle in a situation with all sites at \( Z_0 \). Particles will flow out of the system at any of the two types of boundaries with unit velocity and might annihilate with anti-particles (\( Z_i \rightarrow Z_i + 1 \); \( Z_{i+1} \rightarrow Z_{i+1} - 1 \)). The collision of particles and anti-particles shows that the automaton is not reversible.

In higher dimensions the situation is different since there the situation into which the system naturally evolves does not have an average slope \( \phi = \frac{1}{2} Z \) of \( Z_0 \) but \( \phi < Z_0 \) at which critical fluctuations occur. Let us first define the model in higher dimension \( d \):

\[ Z_i(t+1) = Z_i(t) - 2d \sum_\text{nn} \Theta (Z_i(t) - Z_0) + \sum_\text{nn} \Theta (Z_j(t) - Z_0) \]
\[ (24) \]

where the sum runs over the nearest neighbors of sites \( i \). This corresponds to the smoothing process \( Z_i \rightarrow Z_i - 2d \); \( Z_i \rightarrow Z_i + 1 \) for \( Z_i > Z_0 \) which is like applying a discrete Laplace operator if the variable is above threshold\( Z_0 \).

Again, one can have two boundary conditions:

I) same rule for all sites

II) \( Z_i \rightarrow Z_i - N(Z_i) \) if \( Z_i > Z_0 \)

where \( Z_i \) is a site on the boundary and \( N(Z_j) \) is the number of neighbors that \( Z_i \) has inside the system. Many excitations are possible, a typical particle in two dimensions is given by

\[ Z_i \rightarrow Z_i + 2 \]
\[ Z_{i+1} \rightarrow Z_{i+1} - 1 \]
\[ Z_{i-1} \rightarrow Z_{i-1} - 1 \]

One can easily check that any excitation that falls on the maximal stable configuration where every site has value \( Z_0 \) will produce a sequence of waves which will decrease the value of \( \phi \) if the boundary is of type I.

---

**FIGURE 13**

Clusters of perturbations generated by exciting single sites using boundary condition I taken from Reference 21.

If the system is submitted to many excitations, it finally settles to a critical value \( Z_c \) at which it is critically stable. Because if one excites one site in this critically stable state, it can perturb a cluster of sites of any size. In Figure 13 we see the cluster of all the sites where the variables have been changed due to an excitation at one of the sites. The number \( n(s) \) of clusters of sizes \( s \) per site, i.e. the cluster-size distribution has a powerlaw decay of the form

\[ n(s) \sim s^{-\alpha} \]
\[ (27) \]

and numerically \( \alpha = 2.0 \) in \( d = 2 \) and \( \alpha = 2.33 \) in \( d = 3 \). Since these are dynamical
clusters it is also interesting to know how many time-steps $T$ a given cluster will live. The distribution $D(T)$ of lifetimes also follows a powerlaw

$$D(T) \sim T^{-\sigma}$$

(28)

with $\sigma = 0.43$ in $d = 2$ and $\sigma = 0.92$ in $d = 3$ for boundary condition 1. According to reference 21 one finds different exponents for boundary condition 2. One can consider each time a site diminishes its value of $Z_i$ as an energy dissipation and then show that the resulting power spectrum has so-called $1/f$ noise behavior, namely behaves like $f^{-2}$, where $f$ is the frequency.

It is in fact possible to dress this situation with the usual formalism of critical phenomena. The average slope $\sigma = \langle Z \rangle$ plays the role of the temperature. The order parameter is the total flow $j$ per volume which is just the fraction of sites that have $Z > Z_c$. The field $h$ conjugate to $j$ is the probability to excite a site which for simplicity could be the excitation $Z_i \rightarrow Z_i + 1$. Evidently $\sigma = 0$ if $\phi = \Phi_0$ or if $h = 0$. The susceptibility is then $\chi = \delta j/\delta h$. So one can define critical exponents as usual

$$j \sim (\phi - \Phi_0)^\beta, x \sim (\phi - \Phi_0) \gamma, \chi \sim (\phi - \Phi_0)^{\delta}$$

(29)

For the cluster size distribution one can assume like in percolation

$$\eta(s) \sim s^{-t} F(\phi - \Phi_0)$$

(30)

where $F$ is a scaling function and $t$ and $\sigma$ are critical exponents. The flow response to a small field (excitation) is just a cluster so $\chi$ equals the mean cluster size:

$$\chi = \int \xi^{D} n(s) \psi - \Phi_0 |^{\frac{\xi}{\sigma}}$$

(31)

where $\xi_0$ is the largest cluster. In fact $\xi_0 \sim s^D$ where $D$ is the fractal dimension of a cluster and $\xi$ is the correlation length which goes like $\xi \sim (\phi - \Phi_0)^{-\nu}$. So, since $s^D \sim (\phi - \Phi_0)^{-\nu}$, one has $D = 1/\nu$. From equation (31) also follows $\chi \sim (\xi - \Phi_0)^{\nu}$. For the dynamics one can make the ansatz

$$D(T) \sim T^{-\sigma} G(\phi - \Phi_0) T^{\xi/\nu}$$

(32)

with a scaling function $G$ after some manipulation one can be rewritten as

$$\sigma = 2 - \frac{8}{\nu \xi}.$$  (Note that the $\xi$ of equation 32 is the dynamical exponent, not to be confused with the variable of our model).

These exponents have been obtained numerically in two and three dimensions:

$$\beta = 0.7, \gamma = 1.35, \delta = 1.92, \xi = 1.29 \text{ in } d = 2$$

$$\beta = 0.62, \gamma = 1.7, \delta = 2.97, \xi = 1.7 \text{ in } d = 3$$

(33)

It is also possible to derive mean-field equations for this problem. Let us illustrate this at the one dimensional example ($D = 1$). Let $P_2$ be the probability that a site be in state $2, P_1 = P_2 + P_1$ is the probability for a site to be inactive and $P_A = P_2 + P_1$ the probability for it to be active. Since we want to consider situations close to the critical point let us consider $h \ll 1$ and $P_A \ll 1$; moreover $h$ and $P_2$ should be of the same order and $P_3$ of higher order. The transition from one state to another at a given time step has a certain reaction rate. For the transition from 0 to 1 this rate is for instance $P_0 P_1$ ($1-h$) $P_0 P_1 + P_1$ because an increase can happen due to an active neighbor or due to a spontaneous excitation at this site. The possible transitions in the two lowest orders are

$$0 \xrightarrow{2 \text{ (1-h) } A \ I, \ h \ I^2} 1, \quad 2 \xrightarrow{2 \text{ (1-h) } A \ I, \ h \ I^2} 1$$

$$0 \xrightarrow{(1-h) A^2, 2 \ h \ A \ I} 2, \quad 1 \xrightarrow{(1-h) A^2, 2 \ h \ A \ I} 3$$

(34)

where we used the abbreviation $A = P_A$ and $I = P_I$.

In lowest order only the first three reactions contribute and one gets the equations

$$P_2 = 2 P_0 P_2 + h P_0$$
$$2 P_0 P_2 + h P_0 = 2 P_1 P_2 + h P_1$$
$$2 P_1 P_2 + h P_1 = P_2$$

(35)

For $h = 0$ one immediately obtain $P_1 = P_2 = \frac{1}{2}$ for the critical point.
language \( j = P_3 \) and
\[
\psi = P_1 + 2 P_2 + 3 P_3
\]  
(36)
so that \( \psi = \frac{1}{2} \). To calculate \( j \) one has to consider higher order terms. 

Simplest one can insert Equation 36 into the last equation of (35) in lowest order and obtain
\[
4 P_2^2 + (1 - 2 \psi + 2 \beta) P_2 - h \psi = 0
\]  
(37)

For \( h = 0 \) one finds either \( P_2 = 0 \) or \( j = P_2 = (\psi - \psi_0)/2 \), i.e. \( \beta = 1 \). 

Differentiating Equation 37 with respect to \( h \) and setting \( h = 0 \) one finds
\[
8 P_2 \chi + 2 P_2 + (1 - 2 \psi) \chi - \psi = 0
\]  
(38)
and so \( \chi = 4 | \psi - \psi_0 | \), i.e. \( \gamma = 1 \).

At \( \psi_0 = \frac{1}{2} \) Equation 37 becomes \( 4 P_2^2 + 2 \beta P_2 - h/2 = 0 \) which gives for small \( h : j = P_2 = \sqrt{h/8} \) and hence \( \delta = 2 \).

The relaxation can be obtained by considering that \( P_2 \) is the difference between the two reaction rates, i.e. in lowest order
\[
j = P_2 = 2 P_1 P_2 - P_2 = (2 \psi - 1) P_2
\]  
(39)
The solution of this equation is \( P_2(t) \sim e^{t/\tau} \) with \( \tau^{-1} = 2(\psi_0 - \psi) \), so \( 2\nu = 1 \).

Summing up mean-field gives the exponents
\[
\beta = 1, \gamma = 1, \delta = 2, \nu = 2, \nu = \frac{1}{2}
\]  
(40)

All the scaling relations known from critical phenomena have been verified. 

The mean-field exponents agree with the mean-field exponents of percolation but the exponents in two and three dimensions are different. It has also been argued that the upper critical dimension for this problem is 4 while it is 6 for percolation and even an \( \epsilon \)-expansion in \( \epsilon = 4 - d \) has been given.

5. INHOMOGENEOUS CELLULAR AUTOMATA

In this section and in the next one we want to use a particular definition of a chaotic automaton and for this sake we will define first a distance between two configuration of binary variables \( \{ q_i(t) \} \) and \( \{ p_i(t) \} \):
\[
d(t) = \frac{1}{N} \sum_i | q_i(t) - p_i(t) |
\]  
(41)
called the Hamming distance. It is the fraction of "damaged" sites, i.e. of sites which have different values in the two configuration. \( 1 - d(t) \) is called the "overlap" of the configurations.

We can then distinguish the two situations:

chaotic: \( d(\infty) \rightarrow \text{finite value} \)
\[
d(0) \rightarrow 0
\]  
(42)
frozen: \( d(\infty) \rightarrow 0 \)
\[
d(0) \rightarrow 0
\]

That means that if the initial distance between the configurations is small, the configurations will stay together until infinite time in the frozen case and will go to a finite distance in the chaotic case.

In order to implement the limit to infinitesimal initial distance in practice it is useful to consider three configurations \( \{ q_0 \} \), \( \{ q_1 \} \) and \( \{ q_2 \} \) where \( \{ q_0 \} \) and \( \{ q_1 \} \) differ initially only in \( s \) sites, \( \{ q_0 \} \) and \( \{ q_2 \} \) differ in other \( s \) sites and therefore \( \{ q_0 \} \) and \( \{ q_2 \} \) differ in \( 2s \) sites. Typically one might choose \( s = 5 \). Now the limit of zero initial distance is given by
\[
d(t) = \delta_{q_0}^{\delta_{q_1}}(t) + \delta_{q_1}^{\delta_{q_2}}(t) - \delta_{q_0}^{\delta_{q_2}}(t)
\]  
(43)
In this way one avoids the problem that in a finite system only a finite and not an infinitesimal damage can be implemented.

What we want to study in this section are phase transitions between a frozen and a chaotic phase and for this reason we will study inhomogeneous cellular automata, i.e. automata which do not have the same rule on each cell. A straightforward way to do this is to take a chaotic rule like Q2R and a frozen rule like AND or OR and put on each cell randomly either with probability \( p \) the chaotic rule or with probability \( 1 - p \) the frozen rule. In this way one can obtain all sorts of scenarios. The most important and physically most meaningful model is, however, the Kauffman model as described by S. Kauffman in Reference 1.
The Kauffman model was introduced to describe regulatory systems in biology, particularly for the problem of cell differentiation. In this model one chooses for each cell one of the $2^k$ possible rules randomly. In the original Kauffman model which is an infinite range (i.e. mean-field) model, one chooses also for each cell the $k$ inputs randomly so that the diagram of inputs of Figure 1 is a random graph with fixed $k$. Opposite to this mean-field model we will also consider the finite range model for which one has a regular lattice coordination $k$ and the inputs just come from the nearest neighbors. The rules will be given at the beginning and kept the same for all times (quenched model). Only occasionally, when explicitly stated, we might also consider an annealed model, i.e. a model where the rules are chosen again at each time step, for computational purposes. Such an annealed model would of course not be a deterministic automaton anymore.

A very useful parameter that can be introduced into the model and that will probe the space of possible rules is $p$. Remember that a rule is fully defined by the value it has for the $2^k$ possible input configurations. The Kauffman model at $p$ should mean that for the rules that we use we will choose these $2^k$ values to be "1" with probability $p$ and "0" with probability $1-p$.

Let us first consider the mean-field model which in fact has been studied for a long time. The case $k = \infty$ ($k = N$, the number of cells, and then $N \rightarrow \infty$), when everybody is connected to everybody just gives the so called random map for which it has been shown that the average length $L$ cycles (i.e. the attractors) grows like $2^N$ and the number $N$ of different cycles grows like $N$ with the number $N$ of cells. For $k > 4$ the length $L$ of the cycles seems to grow in the same way as for $k = \infty$. The other known cases are $k = 1$ and $k = 2$ for which $L \sim N^2$, $N \sim \ln N$ and $L \sim N^2$, $N \sim N$ respectively.

The behavior of $d(t)$ in mean-field was given in Reference 25: given $d(t)$ the fraction of sites that at the next time step has the same $k$ inputs in both configurations is $(1 - d(t))^k$ because $1 - d(t)$ is the fraction of sites that have the same value in two configurations. The sites of equal inputs will be the same at $t + 1$. All other sites will be damaged with probability $2p(1 - p)$.

So one has

$$d(t + 1) = 2p(1 - p)(1 - (1 - d(t))^k)$$

In the annealed case at each time the rules are again chosen randomly and so Equation 44 can be iterated. But in the quenched situation we are interested in, when the rules are the same for all times, there will appear a correlation when a site influences the inputs of another site in more than one way. After $T$

Time-steps one site will be influenced by $n = 1 + k + k^2 + \ldots + k^n = \frac{(k^{n+1} - 1)}{(k - 1)}$ sites and the probability that they are all different is

$$\prod_{i=1}^{n} (1 - \frac{1}{N})$$

So one sees that if $N >> n$ the dangerous correlations will not be felt and the quenched model behaves like the annealed model. In the thermodynamic limit Equation 44 is therefore the solution in mean-field.

The stable fixed point $d^*$ of Equation 44 is just the damage $d(\infty)$ at infinite times. One can easily check that for $k \geq 2$

$$d^* = 0 \quad \text{if} \quad p < \frac{1}{2} \quad \text{and} \quad 2\sqrt{2 - \frac{1}{N}}$$

otherwise zero is an unstable fixed point. Since in the above argument the initial distance $d(0)$ did not enter, one has in the case of Equation 45 a particularly strongly frozen phase: whatever the initial distance, $d(\infty)$ is zero with probability one. For $p = \frac{1}{2} \pm \sqrt{2 - \frac{1}{N}}$ one finds, again for any $d(0)$ a finite $d(\infty)$, e.g. $0.382$ for $k = 3$, $p = \frac{1}{2}$ or $0.48$ for $k = 5$, $p = \frac{1}{2}$, and has therefore a chaotic phase.

The frozen phase can be understood via the "forcing" structures (see e.g. Kauffman’s article in Reference 1). One calls a rule forcing from $q_1$ to $q_2$ when having an input with the value $q_1$ enforces the output of the rule to be $q_2$ regardless of the other inputs. An example is OR which is forcing from 1 to 1 or AND which is forcing from 0 to 0. If one connects the output of a forcing rule to the relevant input of another forcing rule, one can construct forcing chains and if in the input diagram like that of Figure 1 one has a loop which is a forcing chain, the cells of this loop will have fixed values. For this reason do forcing structures impose the values in the cells and favour a frozen phase. For $k \leq 2$, $\frac{1}{2}$ of all rules are frozen while for $k = 4$ it is only $\frac{5}{6}$ and the fraction of frozen rules decreases with $k$ at least as $4k^{-1}$. In the frozen phase of an infinite range model, the density of the forcing structures, specially at $k = 2$, seem to be so high that they decouple the different non-forcing cells and effectively the number of cycles become subextensive (i.e. $N^2 \sim N$) which explains how $d(\infty) = 0$, independent on the initial distance.

In the chaotic phase of the mean-field a certain number of quantities of interest can be calculated. So in Reference 26 the distribution $P(m)$ of the local magnetizations $m = \langle q_i \rangle$ (averaged over time) defined as
$$P(n) = \frac{1}{N} \sum_{i=1}^{N} \delta(n - n_i)$$  \hspace{1cm} (46)$$

was obtained. It has singularities at various rational $n$. Also the size
distribution of the basins of attractions of the cycles has been calculated\(^{27}\).
If $\Omega_n$ is the number of initial configurations that lead into cycle $n$, one can
define by $W_n = \Omega_n/2^N$ the weight of this cycle and the quantities $Y_n = \sum_n W_n^k$ as
the moments of the distributions of weights. Numerically $\bar{V}$ and $\bar{V}^2 - (\bar{V})^2$ (the
bar means ensemble average) seem to go to constant values for $k > 2$ when the
size is increased. This effect, which is also found for spin glasses in mean-field,
means that not all basins are of equal size but that there are few big
cycles. Numerically also relations like $\bar{V}_1 = \bar{V}(1 - \bar{V})/2$, which are known from
spin glasses, seem to hold, but an exact calculation for $k = \infty$ gave $\bar{V}_3 = \frac{8}{15}$ and
$\bar{V}_4 = \frac{2}{3}$ which shows that there is a slight discrepancy. Also in the dependence
on the initial distances some discrepancies seem to exist\(^{28}\). Other striking
similarities and small differences between the chaotic phase of the Kauffman
model and spin glasses (both in mean-field) are discussed in various
articles\(^{28-33}\).

Next we will focus on the short-range Kauffman model. Except for the
annealed case which is identical to directed percolation\(^{30}\) no analytical work
has yet been done for this model and one is therefore dependent on simulations.
The most efficient way to implement the Kauffman model on a computer is multi-
spin-coding\(^{31}\). For each of the $j$ possible neighborhoods of site $i$ one defines a
mask $m_j$ which is one if and only if site $i$ has this neighborhood $j$; the update
of site $i$ is performed via
$$O_i(t + 1) = \bigvee_j (O_j \wedge m_j)$$  \hspace{1cm} (47)$$

As for QRH in Equation 19, this is done in parallel for 64 sites through bit by
bit logical operations. Since the process involves $59$ logical functions for
$k = 4$ one has therefore less than one logical function per site.

Numerous simulations have been performed for the short-range Kauffman model
and a short review is given in Reference 32. In one dimension and on the
honeycomb lattice no transition is found and one has only a frozen phase. For
the square lattice the results are presented in Figure 14.

**Figure 14**
Final damage against initial damage above and below $p_c = 0.298$ for the square
lattice. Taken from Reference 30.

Always the phase diagram is symmetric around $p = \frac{1}{2}$ so that for the rest we
will only consider $p \leq \frac{1}{2}$. There are two types of behavior: in the chaotic
behavior ($p = 0.3$) even small initial distances produce a finite final distance
and in the frozen behavior ($p = 0.2$) the final distance decreases linearly
with the initial distance. A similar situation is found for the triangular and the
simple cubic lattice. In contrast to mean-field here the value of the final
distance always depends on the initial distance.

It is possible to cast this situation in the language of critical phenomena:
call $\psi = \lim_{d(0) \rightarrow 0} d(\infty)$ the order parameter and $d(0)$ the field. Then the linear
dependence between $d(\infty)$ and $d(0)$ in the frozen phase corresponds to the Curie
law and one can define a susceptibility as $\chi = \partial d(\infty)/\partial d(0)$. It has been
verified that $\psi$ goes to zero and that $\chi$ diverges at the critical concentration
$p_c$ but the statistical fluctuations have not yet allowed for a reliable
The most accurate method to determine the threshold $p_c$ is the gradient method. One considers a system of linear size $L$ with periodic boundary conditions in the horizontal direction and imposes on each line a different value of $p$ such that on top $p_1 > p_c$ and on the bottom $p_2 < p_c$ and that between the two $p$ vary linearly. The two initial configurations have opposite values on the top line and are identical otherwise, so that one has damage just on the top line. This damage will spread towards the bottom but stop when it comes to lines at $p_c$ because then one comes in the frozen phase (see schematic picture below). One can obtain the average value of $p_c$ at which the front of damage stops and if one extrapolates these values as function of the gradient $V_p = (p_1 - p_2)/L$ to the zero gradient limit one obtains a very precise estimate for $p_c$.

![Diagram of cellular automata and local magnetizations](image)

One can also calculate the distribution $p(m)$ of local magnetizations as defined in Equation 46 and the results is shown in Figure 15. We see that in the frozen phase it consists of many delta functions at rational values of $m$ while in the chaotic phase there is in addition a continuous background. The delta functions can be explained through local periods. Indeed, one can do the same thing as we did for QSR in Figure 11 and this is shown in Figure 16. These local periods evidently produce local magnetizations that in the time average are rational numbers. In Figure 16 we see that the situation is slightly different from the one found for QSR in Figure 11: The clusters of equal period are much more rounded here and it has been observed that they become fractal close to $p_c$. Also in this case a cluster must not be surrounded by “1” but can touch another cluster of incommensurate period. In the chaotic phase in opposition to QSR there are still finite periods (see delta peaks in Figure 15 b) in addition to a spanning cluster of “infinite” period.

![Figure 15: $p(m)$ in the frozen phase (a) and in the chaotic phase (b).](image)

![Figure 16: Local periods in the frozen phase close to $p_c$.](image)
configurations of linear size \( L \) and puts a small damage (difference between the configurations) in the center of the system. In the chaotic phase this damage will grow and reach the boundary of the system after a time \( \tau \) which is proportional to \( L \) leaving behind a number \( N_0 \) of sites that have been damaged during this spreading of the damage and which grows like \( L^\delta \). Now it is interesting to look at the same question exactly at \( p_c \) where the damage just barely spreads. One finds in general

\[
\tau \sim L^\delta \quad \text{and} \quad N_0 \sim L^\delta
\]

and these critical exponents are numerically for the short-ranged Kauffman model \( \delta_k = 1.5 \) and \( \delta_0 = 1.6 \) in \( d = 2 \) and \( \delta_k = 2.2 \) and \( \delta_0 = 1.8 \) in \( d = 3 \). So, \( \delta_k \) becomes larger than one and \( \delta_0 \) less than one. This means that the damage spreads on a fractal subset of space and much slower than a wavefront at the onset of chaos. Note that this effect can only be seen due to the fact that the degrees of freedom are arranged spatially in cellular automata.

When the damage spreads over the system, sites that have been damaged can cure and then be damaged again and so on. If \( n_L \) is the number of times that site \( i \) has been damaged during a fixed period of time \( \tau \), one can define a probability \( p_i \) of being damaged as \( p_i = \frac{n_i}{\tau} \) and consider the moments of its distribution

\[
N_1 = \sum_i p_i L = \tau^{1-q} \quad (49)
\]

which are assumed to scale with time with an exponent \( \delta(q) \). For \( q = 0 \) one obtains just the \( N_0 \) defined above so that according to Equation 48:

\[
\delta(0) = \delta_0 \delta_k. \quad \text{If} \quad p > p_c, \quad \text{i.e.} \quad \text{inside the chaotic phase, one would assume}
\]

\[
\sum_i n_i = N_0 \tau \quad (50)
\]

and therefore

\[
N_1 = \sum_i n_i / \sum_i n_i L = \tau^{1-q} \delta(0)(1-q) = \tau^{1-q} (1-q). \quad \text{This yields}
\]

\[
\delta(q) = \delta(0)(1-q) \quad \text{for} \quad p > p_c \quad (51)
\]

The moments of Equation 49 have been numerically calculated in Reference 36 for
the short-ranged Kauffman model on the square lattice and the $\Phi(q)$ that were obtained are shown in Figure 18 for $p > p_0(b)$ and at $p_0(b)$. We see that the relation of Equation 51 is fulfilled with $\Phi(0) \approx 2$ for $p > p_c$. At the onset of chaos, however, a more complex phenomenon arises, called "multifractality", in which the different moments scale with different exponents. So Equation 50 is not valid anymore at $p_c$ and this can only be due to the fact that there are sites that are substantially more susceptible to be redefined than others. A better understanding of this phenomenon would be useful.

\[ \Phi(q) \text{ as a function of } q \text{ for (a) } p = 0.5 \text{ and (b) } p > p_c \text{ for the square lattice.} \]

Taken from Reference 36.

We have seen for the short-range Kauffman model that at the critical point that separates the frozen phase from the chaotic phase, several particular phenomena occur: percolation of the cluster of unstable sites, stretched exponential growth of the length of the cycles ($1 \sim e^{q}$) with the number of sites $N$, fractal spreading of damage (Equation 48) and multifractality in the damage frequency of the individual sites.

6. DAMAGE SPREADING IN MONTE CARLO

The most popular probabilistic cellular automaton is Monte Carlo dynamics. There are many kinds of Monte Carlo (Metropolis, glider, heat bath, Kawasaki, etc.), but they all are stochastic processes in phase space that lead into thermal equilibrium and stay there. One can study two types of questions: the "non-linear relaxation", i.e., how does the process converge towards equilibrium, and the "linear relaxation", i.e., how does the process move in equilibrium. Roughly speaking, in the first one one studies the transients and in the second the attractors of the dynamics defined by this process.

In view of the last paragraph it seems natural to see Monte Carlo as a cellular automaton and to study if it is chaotic or frozen looking at the development of the distance between two configurations. The main ingredient for this to work is to use for both configurations at each time the same random numbers. So, once the two configurations become the same, they will evidently stay the same. To comply with the requirement of simultaneous update that we need for automata, one has to divide the system in two sublattices as we did it for Q$^2$R and update always one sublattice simultaneously and then the other. If one would update all sites simultaneously without sublattice structure one would instead of a thermal equilibrium with Boltzmann-weight go to a slightly different equilibrium distribution (\[ p \sim \prod_1^{N} \cosh \beta \sum_{\alpha \in \Omega} \])\[17].

Again, two situations can be studied going into equilibrium (non-linear relaxation) and moving in equilibrium (linear relaxation). As we will see, the first one probes the structure of phase space and the second probes the dynamics itself.

Let us first discuss the relaxation into equilibrium as studied in Reference 38. Starting with two configurations and submitting it to the same random numbers one can calculate $P(t)$, the probability that the configurations are still different at time and $< D(t) >$ the distance between the two configurations provided that they are not the same. Note that the distance $d(t)$ we have considered in the last section (Equation 41) corresponds to $d(t) = \sum_{i,j} \delta_{ij} < D(t) >$, since there we averaged over all pairs of initial configurations and not just over the ones that are different. The results found in Reference 38 using heat bath dynamics are shown in Figure 19 for the three dimensional Ising model. In the paramagnetic phase all configurations become identical after some time independent on the initial distance like in the frozen phase of the Kauffman model in mean-field. Below the critical temperature $T_c = 4.2$, $P(t)$ goes to zero when the initial distance goes to zero like in the frozen phase of the short range Kauffman model. If, however, two configurations do not become identical their distance goes to a fixed value $< D(t) >$ that does not depend on $d(0)$. This can be understood by remembering that in the ferromagnetic phase the free energy has two minima and there are therefore two regions of thermal equilibrium in phase space separated by a
distance that is just the spontaneous magnetization. In fact it can be shown that the $<D(t)>$ of Figure 19 equals the spontaneous magnetization. In the paramagnetic phase there is only one connected region of thermal equilibrium in phase space and so all configurations converge together.

![Diagram](image1)

**FIGURE 19**

$P(500)$ and $<D(500)>$ for the 3d Ising model using heat bath dynamics and initial distances $d(0) \rightarrow 0$ (diamonds) $d(0) = 0.5$ (squares) and $d(0) = 1$ (triangles) as a function of temperature. Taken from Reference 38.

We saw that the method of comparing the trajectories of two configurations subjected to the same random number sequence tells us how many disconnected regions of thermal equilibrium or "valleys" there are in phase space. It is natural to try this method now for other models like the three-dimensional Ising spin glass with random couplings of strengths $\pm 1$.

![Diagram](image2)

**FIGURE 20**

$<D(500)>$ for the 3d spin glass as function of temperature. Symbols are defined as in Figure 19 and again heat bath is used. Taken from Reference 38.

The result is shown in Figure 20. $P(500)$ has also been calculated and it is essentially one, independent on $d(0)$ below $T_c = 4.2$. Below $T_g = 1.8$, one believes in the existence of a spin glass phase which has many valleys of all sizes in phase space. The data of Figure 20 show a finite distance at long times even for $d(0) \rightarrow 0$ but the actual value of the distance depends on $d(0)$ like in the chaotic phase of the short-range Kauffman model. This could be understood if one consider that in phase space small valleys are close together and large valleys are farther apart.

Between $T_g$ and the critical temperature $T_c$ of the pure Ising model there is another phase where the final distance is independent of the initial distance $d(0)$ like in the chaotic phase of the Kauffman model in mean-field. This phase, which is known from simulations for its long relaxation times (stretched exponentials) and is sometimes called "Griffiths-phase", can be roughly described as a phase in which some regions in space have an "effective critical temperature" $T_{eff}$ above the temperature considered and some other regions have a $T_{eff}$ below. These $T_{eff}$ separate locally paramagnetic from locally ferromagnetic region and thus two-valley from one-valley situations in phase space. If one accepts this very sloppy picture, it implies that the phase space of the whole system has many equidistant valleys of about the same size and this would explain the behavior of Figure 20.

Concluding, we have found that with the heat bath dynamics one can probe the phase space by starting with two configurations outside equilibrium and one can find all four types of phases that were also found in the Kauffman model. The physical picture that emerges seems consistent for several models. Only recent data for the two-dimensional spin-glass are not really in agreement with this picture and this still needs to be understood.

Let us now discuss the dynamics in equilibrium as done in Reference 40. First, a configuration is thermalized, i.e. one applies Monte Carlo on it until one is in equilibrium. Then, a copy of this configuration is made which differs from the original configuration only very little ($d(0)$ small) and one observes if the two configurations become the same or not after both have been submitted to the same dynamics, i.e. the same random number sequence, for some time. In Figure 21 we see what happens if one applies Glauber dynamics to the two-dimensional Ising model. Apparently the dynamics is chaotic in the paramagnetic phase and frozen in the ferromagnetic phase. This does not contradict our intuition on the effect of temperature on the motion in phase space at low temperatures. Correlations between spins hinder the spreading of damage.
for the Ising model. Correspondingly \( p_{u1} = 1 - 4 \) is the probability that the spin is set to \(-1\). On the computer this dynamics is implemented by choosing a random number \( z \) and defining as the updated spin
\[
\sigma_1' = \text{sign}(\sigma_1 - z)
\]

(53)

Glauber dynamics is a stochastic process that obeys detailed balance. A spin is flipped with probability
\[
p(\text{flip}) = \frac{e^{\Delta E}}{1 + e^{\Delta E}}
\]

(54)

where \( \Delta E = E_{\text{new}} - E_{\text{old}} \) is the difference of the energy \( E_{\text{new}} \) the configuration would have if the spin would be flipped and the energy \( E_{\text{old}} \) of the configuration before any flip. On the computer the update spin is
\[
\sigma_1' = \sigma_1 \text{ sign}(p(\text{flip}) - 2)
\]

(55)

For the Ising model Equation 54 becomes
\[
p(\text{flip}) = \frac{e^{-\beta h_1}}{1 + e^{-\beta h_1}}
\]

(56)

Now one easily verifies that
\[
p(\text{flip}) = \begin{cases} 
1 - (1 - 4) & \text{if } \sigma_1 = 1 \\
(1 - 4) & \text{if } \sigma_1 = -1
\end{cases}
\]

(57)

\[
p(\text{not flip}) = 1 - p(\text{flip}) = \begin{cases} 
(1 - 4) & \text{if } \sigma_1 = 1 \\
1 - (1 - 4) & \text{if } \sigma_1 = -1
\end{cases}
\]

This shows that the probabilities of \( \sigma_1 \) becoming \(+1\) (or \(-1\)) are identical for heat bath and Glauber dynamics. On the computer, however, there is a difference because when inserting Equation 57 into Equation 55 one finds for the Glauber dynamics
\[
\sigma_1' = \text{sign}(\sigma_1 - (1 - 4)) \text{ for } \sigma_1 = 1
\]

(58a)
\[ \sigma_1' = \text{sign}(\sigma_1 - z) \text{ for } \sigma_1 = -1 \] (58b)

This difference does not show up if one simulates just one configuration because the probability to have \( Z > 1 - \frac{\mathcal{t}}{2} \) (Equation 58a) is the same as the probability to have \( Z < \frac{\mathcal{t}}{2} \) (Equation 58b). Note that \( Z \) is homogeneously distributed between zero and one. If, however, one has two configurations that use the same random number \( Z \), as it is our case, and \( \sigma_1 = +1 \) for one configuration and \( \sigma_1 = -1 \) for the other configuration but the same \( \mathcal{t} \), then for heat bath dynamics one gets for both the same result and for Glauber dynamics not necessarily. Actually one finds at infinite temperature \( (\mathcal{t} = 0, \mathcal{t} = \frac{1}{2}) \) that in Glauber dynamics Equations. 58a and 58b do just the opposite.

It is now clear why the two dynamics give different results but it would be nice to understand if there is a general principle that tells how a given dynamics will behave, and it would be in particular interesting to know how other dynamics like Metropolis or Kawasaki behave.

Let us next present some precise results that have been derived for the damage. Suppose we have two configurations A and B. In B we fix at the origin the site to be always \( \sigma_B^0 = -1 \) and in A there is no restriction. This site already induces a damage (when due to the Monte Carlo simulation \( \sigma_A^0 \) happens to be +1). Let us in the following also switch to variables \( \pi_1 = \frac{1}{2}(1 - \sigma_1) \), i.e. \( \pi_1 = 0, 1 \). And let us distinguish two types of damage at a given site \( i \): a damage where \( \sigma_i^A = +1 \) and \( \sigma_i^B = -1 \) which should occur with probability \( d_i^- \) and a damage where \( \sigma_i^A = -1 \) and \( \sigma_i^B = +1 \) which should occur with probability \( d_i^+ \). So we have

\[ d_i^- = \langle \pi_i^A (1 - \pi_i^B) \rangle \text{ and } d_i^+ = \langle (1 - \pi_i^A) \pi_i^B \rangle \] (59)

where \( \langle \ldots \rangle \) is either a temporal or a configurational average (suppose ergodicity). Let us now calculate the quantity

\[ \Gamma_1 = d_i^- - d_i^+ = \langle \pi_i^A \rangle - \langle \pi_i^B \rangle \] (60)

where in the second equality we used Equation 59. Let us consider an Ising system \( \{ \pi_1 \} \) without any restrictions (like for instance system A). Then one can write

\[ \langle \pi_i^A \rangle = \langle \pi_i \rangle \quad \text{and} \quad \langle \pi_i^B \rangle = \frac{\langle \pi_1 (1 - \pi_0) \rangle}{1 - \pi_0} \] (61)

where the second equation is just the conditional probability with the condition \( \pi_0 = 0 \). Inserting in Equation 60 we obtain

\[ \Gamma_1 = \frac{\langle \pi_1 \pi_0 \rangle - \langle \pi_1 \pi_1 \rangle}{1 - \langle \pi_0 \rangle} = \frac{\langle \pi_1 \rangle \langle \pi_0 \rangle - \langle \pi_0 \rangle^2}{2(1 - \langle \pi_0 \rangle)} \] (62)

where we used the spatial homogeneity of the magnetization \( m = \langle \sigma_1 \rangle = \langle \pi_0 \rangle \). Defining as usual the two-point correlation function \( C_{ij} \) as

\[ C_{ij} = \langle \sigma_i \sigma_j \rangle = -\langle \sigma_0 \rangle^2 \] (63)

we have therefore obtained a relation

\[ \Gamma_1 = \frac{C_{ij}}{2(1 - m)} \] (64)

for the difference between the two types of damage and usual thermodynamic functions like the correlation function \( C_{ij} \) and the magnetization. This relation is valid for any model of binary variables, any dynamics, any system size, at any time and also without using the same random numbers.

If we consider now the Ising model using heat bath dynamics and the same random numbers, we see that at \( t = 0 \)

\[ \mathcal{t}^A = \mathcal{t}^B \] (65)

for any \( i \) where \( \mathcal{t} \) was defined in Equation 52. If at the next time step \( Z > \mathcal{t}^B \), i.e. if \( \sigma_i^B \) becomes -1, then, because of Equation 65, also \( \sigma_i^A \) must become -1. So again for the next time step Equation 65 will be valid and by induction we see that a damage of the type \( \sigma_i^A = -1 \), \( \sigma_i^B = +1 \) can never occur. Thus in this case \( d_i^- = 0 \) and therefore we have for the total damage at site \( i \) defined of course as \( d_i = d_i^- + d_i^+ \) because of Equation 64:

\[ d_i = \frac{C_{ii}}{2(1 - m)} \] (66)

So, the damage that we usually measure is related to \( C_{ii} \) and \( m \). Consequently, also the susceptibility \( \chi = \sum_i C_{ii} \) is related to the total damage \( d = \sum_i d_i \) of
the system via

\[ \chi = 2(1 - m) d \]  \hspace{1cm} (67)

In Figure 22 we see that the correlation function calculated via Equation 66 is less susceptible to statistical fluctuations than when calculated in the usual way. Note that the open triangles for \( r > 12 \) in Figure 22 have error bars of the same size as their value, i.e. of the order of \( 10^{-3} \) and only therefore do they deviate so much from the circles which are much more precise. So, as a byproduct of Equation 66 we have found a numerically very efficient method of calculating correlation functions.

![Graph showing correlation function](image)

**FIGURE 22**

Let us define the correlation function \( G(r) = \sum_{l=-\infty}^{\infty} C_{0l} \) where the sum goes over all sites that have a distance \( r \) from the origin. We show \( G(r) \) calculated directly (triangles) and via Equation 66 (circles) for two different temperatures in the paramagnetic phase. Taken from Reference 36.

A nice application of relation 66 is shown in Figure 23 which shows all the sites that are damaged at a given time at two different times. These clusters have a fractal dimension of \( d - \beta/\nu \) where \( \beta \) and \( \nu \) are the exponents of the two dimensional Ising model. These clusters are a direct visualization of the fluctuation of the magnetization induced from one site and a superposition of them should yield the typical coalescence clusters seen in the density fluctuations of a binary mixture.

**FIGURE 23**

Clusters of damaged sites if in one system the central site is fixed to -1. Taken from Reference 44.

Finally let us mention that one can also study, using the same methods as described here for Monte Carlo, deterministic cellular automata with noise. In Reference 45 this has been done for the Kauffman model. An each site one defines an energy \( \sigma_{1} f_{1} \) where \( f_{1} \) is the rule that determines the value of \( \sigma_{1} \) (see Equation 1). Like in the Ising model, minimizing this energy, i.e. at low temperature, one will obtain \( \sigma_{1} = f_{1} \), the desired deterministic rule. We see that \( f_{1} \) plays the role of a local field analogous to the \( h_{1} \) that is used in the heat both dynamics of Equation 52. So in this case one defines the probability for \( \sigma_{1} \) to be one as

\[ P_{1} = \frac{e^{\beta_{1}}}{1 + e^{\beta_{1}}} \]  \hspace{1cm} (68)

Now again one investigates the distance of two initially close configurations that are submitted to the dynamics of

\[ \sigma_{1}' = \text{sign}(P_{1} - Z) \]  \hspace{1cm} (69)

using the same random numbers for both configurations. In this way again one
can distinguish a frozen and a chaotic phase. The phase diagram can be seen in Figure 24. There are results from a simulation for the square lattice and an exact line calculated on a so-called Bethe lattice, i.e., a mean field calculation. One sees that the temperature-driven noise tends to reinforce the frozen phase since it tends to make the system less sensitive to the initial configuration. This procedure can of course be executed for any deterministic automaton.

![Phase Diagram](image)

**Figure 24**

Phase diagram in the $e^\beta - p$ plane for the Kauffman model. The crosses are for the square lattice and the full line for the Bethe lattice. Taken from Reference 45.

7. OUTLOOK

The interesting questions on cellular automata in physics, even in statistical physics, are by far not exhausted in the above course. An interesting development that is worth mentioning are the probabilistic automata considered by Dornay and Kinzel in which the values of the rules are assigned with a certain probability and which can be mapped to certain Ising and percolation models and for which powerful methods from statistical physics like transfer matrices can be applied.

Another automata that are even potentially of technological interest are lattice gas automata which are very well reviewed in Reference 47. These automata pretend to simulate hydrodynamics via particle motion with simple collision rules. This subject is very active today but also so complex and long that it would fill a whole course on its own.

I hope the reader got an impression on the variety and complexity of the calculations that can be performed on cellular automata in the context of statistical physics and on the interesting new phenomena in dynamics that can be observed through them.

ACKNOWLEDGMENT

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Nonlinear Phenomena in Complex Systems

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PREFACE

The Workshop on Non-linear Phenomena, and the Third Argentine Workshop on Non-Equilibrium Statistical Mechanics and Non-linear Physics were held in Mar del Plata, a city on the seashore of Buenos Aires Province, from 1–14 November, 1988. The series of Argentine workshops was known as “MEDYFINOL” conferences (a shortened form for Mecánica Estadística del Desequilibrio y Física no Lineal). During 1986, 1987, 1988 (and also in 1989) I had the pleasure of sharing the direction of these meetings with Prof. Angel Piastino (Dept. of Physics, Universidad Nacional de La Plata), Dra. Susana Hernández (Dept. of Physics, Universidad de Buenos Aires), and Dr. Dino Otero (Laboratorio de Física-Matemática, Comisión Nacional de Energía Atómica).

Our main purpose was to provide a forum for discussing recent developments and communicating the latest work of the researchers, as well as to give intensive training to advanced theoretical physics students in the pertinent topics.

This year, our meeting was included in the activities supported by the Office of External Activities of the International Center for Theoretical Physics (ICTP). On behalf of the participants, the conference directors would like to acknowledge, with gratitude, the generous financial support of Professor Abdus Salam, which allowed us to have three distinguished lecturers — John Clark, Rudolf Friederich, and Hans Herrmann. We would also like to acknowledge the valuable support of the Comisión de Investigaciones Científicas de la Provincia de Buenos Aires and the Secretaría de Ciencia y Técnica de la Nación as well as the sponsorship given by the Comisión Nacional de Investigaciones Espaciales, the Universidad de Buenos Aires, the Universidad Nacional de La Plata, the Universidad Tecnológica Nacional, UNESCO, the Latin American Centre of Physics (CLAF), and the Comisión Nacional de Energía Atómica. To the Universidad Nacional de Mar del Plata, the host institution of the meeting, and to the local organizers, Lic. Juan R. Sanchez, Lic. Constancio M. Arizmendi, and Mr. Alberto H. Rizzo, we express our gratitude for their kindness and efficiency.

The success of the conference was partly due to the active support of three members of my research group, Dr. Jorge Aliaga, Lic. Gustavo Crespo, and Lic. Luis Irastorza, who helped with the organization and carried out a wide variety of tasks. They courteously helped me to prepare these proceedings, by gathering, reading, and undertaking the onerous task of typing some of the conference material. To them, my special thanks.
INTRODUCTION

This book provides a thorough treatment of neural networks, cellular-automata, and synergetics, to illustrate three different approaches to non-linear phenomena in complex systems. These topics are of special interest to physicists working in the fields of statistical mechanics and dynamical systems. The chapters are written with a high degree of sophistication and include the refinements necessary to work with the complexity of real systems. Recent research developments in these areas are included as well.

Neural networks are currently arousing the interest of scientists in a number of disciplines: as models for understanding how the brain works; as a way to exploit the principles of natural intelligence for practical purposes; as examples of complex statistical systems; as modern computational structures; and as devices for intelligent pattern recognition. Dissipative non-linear dynamical systems of interconnected neuronal elements are treated here by providing a broad survey of the modelling of neural phenomena.

Cellular automata play a central role in the understanding of complexity and its origins, since traditional mathematical models and methods do not seem to be well suited to study complex systems. Physics', biology's and other fields' experiments that show complex behavior can be successfully approached with these models which are simply constructed, but contain the essential mathematical features that can reproduce experimental data. Cellular automata can also be viewed as computational models and they are likely to be particularly important to the implementation of future generations of parallel computers.

The essence of synergetics lies in the study of the cooperation of the many subsystems which compose a complex system. This cooperation gives rise to spatial, temporal and functional structures. Special attention is given to those systems in which these structures appear in a self-organized fashion. The principles governing these self-organization processes can be studied, irrespective of the nature of the component subsystems. The applications of synergetics ranges from physics to sociology, making the importance of its basic concepts and mathematical approach evident.

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