The Hunt for Universality in Fracture

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Fracture of disordered media is a technologically important subject for which, up to now, not very much has been theoretically understood. Materials as diverse as concrete, textiles, glass, paper and soil can rupture if they are subjected to an external strain and we want to ask if there might nevertheless be some universal laws governing the size dependence of the breaking, independent of the elastic properties of the material and of the type of disorder.

Let us consider a finite regular lattice of size \( L^3 \), periodic boundary conditions in the horizontal directions, and fixed plates on the top and the bottom. On these plates the external strain will be applied. Each bond of the lattice should model the reality on a mesoscopic level and is supposed to be ideally fragile; i.e. to have a linear elastic dependence between force and displacement up to a certain threshold displacement \( \lambda_c \) (see inset of Fig. 1). The thresholds are randomly distributed according to some probability distribution \( P(\lambda_c) \). As the external strain is increased, one can watch bonds breaking one by one until the system breaks apart altogether.

Three different models, corresponding to different physical situations, have been investigated. The scalar model is equivalent to a network of electrical fuses, if one replaces force by current and displacement by voltage. In this model on each site there is only one scalar variable, namely the electrical potential. In the central force model, which for stability reasons must be implemented on a triangular lattice, the bonds are springs that can freely rotate around the sites and on each site is a d-dimensional vector as variable, namely the displacement. The beam model describes full bond-bending elasticity. On each site one has a rigid cross which can be rotated, in 2d, by one angle \( \theta \) which represents an additional variable for each site. The bonds are elastic beams soldered at the crosses at the adjacent sites and their elastic energy is, in 2d, the sum of the contributions from elongation, shear and flexion. A beam breaks if

\[
(F/t_p)^2 + \max(|M|, |N|)/t_m > 1,
\]  

where \( F \) is the elongation force and \( M \) and \( N \) are the moments that act on the sites at its extremities. Thus, two randomly chosen thresholds, \( t_p \) and \( t_m \), are needed to describe the two dominant mechanisms that break a beam in Eq.(1): elongation and flexion.

132

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In one dimension the model is trivial and in higher dimensions various approximations have been proposed\(^2\). For simplicity let us consider the scalar model. On the bond at the tip of a linear crack of \(k\) adjacent vertical bonds, the voltage drop is enhanced by a factor \(\alpha_1\) which, for \(L \to \infty\) and no other cracks close, is \(\alpha_1 = \frac{h}{\pi}\) for the square lattice and goes like \(\alpha_1 \propto 1 + c \frac{L^{1/2}}{(4^2-1)}\) for large \(L\). If one considers that the distribution of voltage drop thresholds \(P_{v_c}\) has a lower cutoff \(v_c\) and an upper cutoff \(v_c = v_c + W\), one sees that beyond a critical crack length \(k'\) given by \(v_c \alpha_1 \geq v_c\), cracks are certainly unstable. But even if one considers only cracks of size unity, after an average number \(n_1\) of such cracks given by

\[
(v_c + n_1 W/L^2) \alpha_1 = v_c + W/(2n_1 + 1) \tag{2}
\]

one of these cracks will become unstable. Eq.\((2)\) implies that if \(v_c \neq 0\) only a finite number of bonds \(n_1\) must be broken before one crack of length \(k'\) appears, which will then break the system apart. For \(v_c = 0\) one finds that \(n_1\) grows at least like \(L^{4/2}\). One can on the other hand also ask which is the typically largest crack one would find if bonds were broken in an uncorrelated way\(^9\). One finds that its length \(k'\) would depend on \(L\) like \(k' \propto \ln L\). Under this assumption the system becomes unstable for \(k' = k''\), i.e., for an unenhanced local potential drop of \(v \propto v_c/((\ln L)^{1/2}(4-1))\). Since the cutting of bonds is not uncorrelated, \(n_1\) cannot grow faster than \(L^2/((\ln L)^{1/2}(4-1))\).

To see how \(n_1\) really grows with \(L\), and also to study other properties, we simulated in two dimensions all three models using conjugate gradient relaxation techniques\(^10\). Typically we averaged over 50,000 samples for \(L = 4\) and less than 10 samples for \(L = 64\). We looked at power-law distributions \(P(\lambda) \propto \lambda^{-\alpha}\) for \(-1 < \alpha < 0.8\) and, in the scalar case\(^3\), also at Weibull distributions \(P(\lambda) \propto \lambda^{\alpha-1} e^{-(\lambda/\theta)^\alpha}\) for \(2 < \alpha < 10\).

After having removed a bond we recalculate the equilibrium (minimum of total elastic energy) and find the then weakest bond. This bond is then removed and everything is started over again. We monitor the external force \(f\) (current) and displacement \(\lambda\) (voltage) needed to break each bond. The dependence between the two is the breaking characteristic shown in Fig. 1. As opposed to the characteristics of the individual bonds we see that the total system is ductile. We distinguish three regimes: in the beginning the curve is linear and, as seen in Fig. 1, one has a scaling behaviour: \(f = L^\alpha \Phi(\lambda L^{-2})\), where \(\alpha = \beta \approx 0.75\) is found numerically. For the number \(n\) of bonds cut we also verified a law \(n = L^\beta \Phi(\lambda L^{-2})\) with \(\gamma \approx 1.7\). This regime is dominated by disorder.

The second regime is given by the maximum of the curves in Fig. 1. The number \(n_0\) of bonds that have been broken when this maximum is reached scales like \(n_0 \propto L^{1.7}\), which is stronger than \(L\) and weaker than \(L^2/\ln L\) as predicted by the approximations. When the disorder goes to zero, i.e., \(x \to \infty\) or \(m \to \infty\), one crosses over to a constant \(n_0\) in agreement with the result of Eq.\((2)\). For the scalar case we also analyzed\(^3\) the moments of the distribution of local voltages.
Fig. 1: Breaking characteristics scaled by $L^{0.75}$ for the beam model with $x = 0.5$ (see Ref. 7). The (smoothed) data come from sizes $L = 4, 8, 16$ and $32$. Inset: characteristic of a single beam.

Fig. 2: Moments $M_0(0), m_1(x), m_2(\delta), m_6(\theta)$ and $m_9(\psi)$ as a function of $L$ for the scalar model and Weibull distribution with $m = 2$ taken from Ref. 3.
and found that they all scale with one exponent, i.e. we have gap scaling and no multifractality\textsuperscript{11}.

After the maximum, one finally has the catastrophic regime, in which each time less force is needed to break the next bond and where actually only a very few unstable cracks grow together virtually on straight lines. This regime has very strong statistical fluctuations and the number \( n_c \) of bonds cut to finally break the system apart also goes like \( n_c \approx L^{1.7} \). For vanishing disorder this crosses over to \( n_c \approx L \). The voltage (or force) distribution \( n(v) \) of the system before the last bond is cut is multifractal, as can be seen by plotting in Fig. 2 the moments \( m_q = \langle \frac{\Delta v^q}{\Delta v} \rangle \) with \( m_q = \sum v^q n(v) \) and finding that their dimension continuously varies between \( m_0 \approx L^2 \) and \( m_\infty \approx L^9 \).

We found that the number of bonds cut scales in all regimes, for all three models and any non-vanishing disorder like \( L^{1.7} \), with an uncertainty in the exponent of less than 10\%. We conjecture that this number equals the fractal dimension of DLA \textsuperscript{12}. The force (or voltage) distributions are multifractal at the end of the breaking but not at the maximum of the applied external force (or voltage).

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References