Disorder-Induced Nonlinear Conductivity.

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Abstract. – We consider an electrical network of resistors in two dimensions. Each resistor has a threshold value for the potential drop below which it becomes an insulator. If the thresholds are randomly distributed, the current flowing through the total network is not proportional to the external voltage. A new nonlinear relation between current and voltage appears.

Disorder can be responsible for curious phenomena. Famous examples are diluted magnets, localization or anomalous diffusion. We want to report in this letter on another novel phenomenon induced by disorder: the appearance of an unexpected power law regime in the voltage-current characteristic of a network consisting of resistors which individually display piecewise linear regimes separated by randomly distributed thresholds (fig. 1a)).

Electrical devices having such randomly distributed thresholds do exist in electrochemistry or, for instance, in the nervous system. Another direct application of our result appears in hydrodynamics of porous media: namely in the problem of a Bingham fluid[1], if one replaces the voltage drop to the pressure drop and the current to the flow rate. Individual channels can have characteristics like that of fig. 1a) and, therefore, the random medium should show effects similar to those reported in this letter.

We consider a regular network of size $L \times L$ and on each bond we place an electrical device having the characteristic shown in fig. 1a). If the absolute value of the voltage drop $v$ across the bond is less than a threshold value $v_g$ no current flows, i.e. the device behaves like an insulator. For $|v| > v_g$, one has a usual linear characteristic, i.e.

$$i = \sigma (v - v_g), \quad \text{for } v > v_g,$$

$$i = \sigma (v + v_g), \quad \text{for } v < v_g$$

(see fig. 1a)). The conductance $\sigma$ is the same for all bonds. If also the thresholds $v_g$ of all bonds are equal, then the voltage-current characteristic of the whole lattice will be the same as that of an individual bond but with a threshold value $V_g \propto L \cdot v_g$. 
Now we introduce disorder. The individual thresholds $v_g$ are chosen randomly from a uniform distribution between 0 and 1. We shall now analyse how this does change the voltage-current ($V$-$I$) characteristic of the network. ($V =$ external voltage; $I =$ total current flowing through the network.) There will be a threshold value $V_g$ so that for $|V| < V_g$ no current flows. $V_g$ is given by

$$V_g = \min (\Sigma v_g) ,$$

where the minimum is taken over all paths between the two electrodes at which the external voltage is applied and the sum includes all bonds within one path. If $V$ is large enough, namely if it is larger than a threshold $V_g$, all individual bonds are conducting and $V$ varies linearly with $I$. In this regime, the conductance of the lattice, $\Sigma$, will be that of a regular lattice of linear resistors of conductance $\sigma$. But what will be the relation between $I$ and $V$, for $V_g < V < V_g$? This is the main question we address in this letter.

We consider a finite square lattice, tilted by 45° (fig. 1b). On top and bottom, we place our electrodes (equipotentials) so that they lie on diagonals of the lattice. In the horizontal direction, we implement periodic boundary conditions. We set $\sigma = 1$. For a fixed total current we calculate the voltage distribution on the sites of the lattice using a standard Gauss-Seidel overrelaxation technique. We start with a large current (so that $|V| > V_g$) and decrease it then by successive factors of two. Each time we calculate the external voltage drop needed to maintain the current and take out all bonds that carry no current. (This has no effect on the rest of the computation, since those bonds will certainly remain insulators also for smaller currents.) On the other hand, $V_g$ can be calculated exactly through a searching algorithm using eq. (1). We find numerically a relation $V_g = (0.22 \pm 0.02) L$.

Figure 2 shows a log-log plot of $\langle (V - V_g) \rangle$ vs. $I$, for various lattice sizes $L \times L$. The brackets $\langle \ldots \rangle$ stand for an average over different configurations. Shown are the data for $100 \times 100$, $20 \times 20$ and $5 \times 5$ with, respectively, 2, 40 and 60 samples. Clearly for a given $L$, the curves exhibit three different regimes with a linear relation between $\log (I)$ and $\log (V - V_g)$:

i) Large values of $I$: this corresponds to a lattice with all bonds conducting ($V > V_g$) and thus, as seen previously, the data for all sizes will reach an asymptotic behaviour given by

$$\log (V - V_g) = \log (I) - \log (\Sigma) .$$

For our tilted square lattice $\Sigma = 2\sigma$. This exact asymptotic line is shown in fig. 2.
Fig. 2. – Log-log plot of $V - V_g$ against the current $I$ for different sizes $L$ of networks: ▽ 5, ○ 20, ▲ 100. Three guides to the eye are drawn: a line of slope one for small currents, a line of slope 1/2 in the intermediate range and the exact asymptotic behaviour at large currents.

ii) Intermediate values of $I$: transition regime for which the number of conducting paths changes. The data follow a power-law relation:

$$V - V_g \propto I^\alpha.$$ 

The value of $\alpha$ is larger than 1/2 for small system sizes ($L \leq 10$) and converges to 0.50 ± 0.02 for $L \geq 20$. The extent of validity of the latter relation increases with $L$. For $L = 100$, the relation is verified over more than three decades.

iii) Small values of $I$: we defined $V_g$ as the voltage at which a first path becomes conducting. In a finite system, in order to have a second path, one has to increase the voltage by a finite amount $\Delta V$. Thus for $V_g < V < V_g + \Delta V$ one expects

$$V = V_g + rI,$$

where $r$ is the resistance of the first path, i.e. $r = \lambda/\sigma$, where $\lambda$ is the length of this path. Therefore, for a given size $L$, the data should fall on an asymptotic line, when $I$ tends to zero, given by

$$\log(V - V_g) = \log(I) - \log(\sigma/\lambda).$$

Using this equation, we can extract from fig. 2 the value of $\lambda$ and we find, within our error bars,

$$\lambda = L.$$

In addition this implies that this last regime is actually a finite-size effect, because the gap $\Delta V$ will tend to zero as $L^{-1}$ when the lattice size goes to infinity. The log-log plot of fig. 2 shows indeed that the domain of validity of the second regime increases with the size of the system.

We propose a mean-field argument to understand the transition regime: let us consider an external voltage $V$ ($V_g < V < V_g$) and a small increase $V + dV$. The increase in the voltage drop felt by every bond in the lattice is assumed to be proportional to $dV$. Since the thresholds, $v_g$, of the bonds were distributed uniformly, the number of bonds, $dn$, that start
carrying current is also proportional to dV. So also the conductance of the lattice increases by an amount dΣ. Neglecting the correlations among the conducting bonds, we can use a classical self-consistent result [2], which gives dΣ ∝ dn, thus dΣ ∝ dV. Using
\[ dI = \Sigma dV, \]
we obtain
\[ I \propto (V - V_g)^2. \]

This is in excellent agreement with our numerical result.

The existence of the transition regime where a novel power law relation, eq. (2), holds is also encountered for more general types of voltage-current characteristics in the individual bonds. If instead of a linear dependence of \( i \) vs. \( (v - v_g) \) (fig. 1a) at \( |v| > v_g \), we consider a quadratic relation
\[ i = \tau (v - v_g)^2, \]
the same relaxation method can be applied. Figure 3 shows, for different lattice sizes \( L \), the three regimes found previously in fig. 2. The large and small values of \( I \) are of the type of eq. (3): \( I \propto (V - V_g)^2 \), whereas the intermediate regime is well described by a power law like that of eq. (2) but where the exponent is \( \alpha = 0.37 \pm 0.04 \), clearly smaller than 1/2. In this latter case, the mean-field argument given previously cannot apply.

\[ \text{Fig. 3. – Log-log plot of } V - V_g \text{ against } I \text{ for different sizes } L \text{ of networks of nonlinear resistors: } \triangledown 5, \times 10, \circ 20. (\text{For every bond the current is proportional to the square of the voltage drop above the threshold.}) \]

We have considered only thresholds distributed between 0 and 1. We note that any uniform distribution on an interval will give the same exponent; only the size of the new nonlinear regime will depend on the width of the distribution. Other types of distribution might lead to different behaviours as encountered, e.g., in ref. [3].

A related, though more complex, problem can be found in a mechanical analogue: the elasticity of a piling of cylinders with fluctuating radii. It has been found, both experimentally [4] and numerically [5], that if the applied force is less than a certain value, then a stress-strain relation is found compatible with a power law (analogous to eq. (2)) behaviour with a large exponent.

Nonlinear random resistor networks have been studied previously in the context of percolation [6]. Also the nonlinear corrections to the conductivity close to the percolation
threshold have been investigated [7]. Furthermore, there is a recent effort to understand fracture through resistors with thresholds (fuses) [3]. Our finding, the appearance of a nonlinear regime due to random thresholds, should be relevant in those cases and we think that it would be fruitful to develop these ideas. We also want to emphasize that it should be possible to see the effect we described in this letter directly in experiments.

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REFERENCES


