ON FINITE-SIZE SCALING OF THE ORDER PARAMETER IN PERCOLATION

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Finite-size scaling of the order parameter $O$ in the disordered phase is presented for percolation in 2d and 3d and compared to the behavior of the magnetization of the Ising model. We find that the order parameter $O$ grows logarithmically with the system size $L$ for $p < p_c$ and thus the analogy with the Ising model is not complete. Furthermore we find evidence for scaling behavior of the type $O \sim L^{-\gamma/\nu} (p_c - p)^{\beta - \gamma}$, where the exponent $\gamma$ is not $\nu d/2$ as in the Ising case.

The order parameter $O$ of a phase transition — e.g., the magnetization $M$ of the Ising model or the fraction of sites in the infinite cluster $P_\infty$ in percolation — is finite in the ordered phase and zero in the disordered phase if one takes the thermodynamic limit. In a finite system, however, the order parameter is usually non-zero everywhere (we define the order parameter of the Ising model for the finite system as $M \sim (\langle M^2 \rangle)^{1/2}$ [2] and the order parameter $P_\infty$ for the finite system in percolation as the number of sites which belong to the largest cluster [1]). Finite-size scaling phenomenologically describes how the order parameter of a finite system tends to its value in the limit of large system sizes $L \to \infty$:

$$O = L^{-\beta/\nu} f[\epsilon L^{1/\nu}] .$$

(1)

Here $\epsilon$ is the distance from the critical point and $f(x)$ is a scaling function with $f(x) \sim x^\beta$ for $x \to \infty$ in the ordered phase. In the infinite system $\beta$ is the exponent of $O$ in the ordered phase $O \sim \epsilon^\beta$. Relation (1) can be numerically tested by plotting $OL^{\beta/\nu}$ against $x = \epsilon L^{1/\nu}$. All points for different $\epsilon$ and $L$ should collapse on a single curve $f(x)$ [11]. In the ordered phase one expects a plot made double-logarithmically to have a slope $\beta$ for large $x$. In the disordered phase one finds numerically that eq. (1) also holds but the slope in the log-log plot is negative [4,12]. In ref. [4] such a plot is shown for the 2d Ising model and for $T > T_c$ the scaling function seems to scale as $f(x) \sim x^{\beta - \gamma}$ for $x \to \infty$ where $\gamma = 1$. The fact that $\gamma = 1$ for the 2d Ising model agrees well with an argument from block distribution functions [2] that for fixed $T > T_c$ the magnetization of the $d$-dimensional Ising model goes to zero as $L^{-d/2}$ so that the slope in fig. 3 of ref. [4] is really $\beta - \gamma = \beta - 1/2d\nu$.

We analyze Monte Carlo data for percolation by making a finite-size scaling log-log plot of $PL^{\beta/\nu}$ against $(p - p_c) L^{1/\nu}$ on a triangular lattice (fig. 1) and simple cubic lattice (fig. 2) for free boundary conditions. In 2d we use the exact values $p_c = 0.5$, $\nu = 4/3$ and $\beta = 5/36$ [5], in 3d we use $p_c = 0.3117$, $\nu = 0.88$ and $\beta/\nu = 0.5$ [6–8]. We see that apparently all the

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Fig. 1. Finite-size scaling plot of the fraction of sites in the largest cluster \( P_{\infty} \) for percolation on triangular lattices of size \( L \times L \) for concentrations \( p \) above and below \( p_c = 0.5 \). The data shown come from Monte Carlo simulations with free boundaries based on statistics of up to 50 samples. The values of exponents used are \( \nu = 4/3, \beta = 5/36 \) [5]. For \( p > p_c \) the solid line corresponds to the expected slope \( \beta \).

Fig. 2. Finite-size scaling plot of \( P_{\infty} \) for percolation on simple cubic lattices of size \( L \times L \times L \) for concentrations \( p \) above and below \( p_c = 0.3117 \) [7,8]. The Monte Carlo statistics are up to 50 samples. The values of exponents used are \( \nu = 0.88, \beta/\nu = 0.5 \) [6,8]. For \( p > p_c \) the solid line corresponds to the expected slope \( \beta \).

Data collapse on one curve so that finite size scaling is valid. In the ordered phase, \( p > p_c \), the slope in figs. 1 and 2 is \( \beta \) as expected. In the disordered phase, \( p < p_c \), we find that the assumption of eq. (1) is valid, but for 2d the slope is \( \beta - y = -2.4 \pm 0.6 \), and for 3d \( \beta - y = -1.7 \pm 0.3 \), thus yielding \( y/\nu = 1.9 \pm 0.4 \) for 2d and \( y/\nu = 2.5 \pm 0.3 \) for 3d. Thus, clearly \( y/\nu \) is different from the value \( 1/2d \) of the Ising model. Instead, the data suggest that \( y/\nu = d \) in 2d in agreement with the notion that the size of the finite largest cluster for \( p < p_c \) is essentially independent of \( L \) for \( L \to \infty \) and thus, trivially, \( P_{\infty} L^{\beta/\nu} \sim L^{\beta/\nu - d} \). In 3d this speculation seems numerically less plausible, and this is supported by the fact that also the slope obtained from the finite-size scaling plot for a kinetic percolation problem on fig. 7 of ref. [9] yields \( y/\nu \approx 2.3 \), which suggests that in 3d \( y/\nu < d \). But the figures show a curvature for large \( \epsilon L^{1/\nu} \) so that there is some uncertainty in the extraction of the exponents.

A more detailed examination of the large \( L \) limit reveals that if we fix \( p \) (with \( p < p_c \)), \( O \) behaves as [1]

\[ O \sim L^{-d} \log L. \]

(2)

This is due to the fact that below \( p_c \) the asymptotic animal behavior dominates [1] for extremely large clusters

\[ n_s \sim \lambda^s s^{-\theta}, \]

(3)

where \( \lambda \) is a growth parameter and depends on \( p \), and
\( \theta \) is a known critical exponent. Let us choose \( s^* \) so that only a fixed number, \( N \), of clusters have a size \( s \) larger than \( s^* \). Then \( s^* \) is proportional to the size of the largest cluster \([8]\). Thus

\[
N = L^d \sum_{s \geq s^*} n_s \sim L^d \lambda s^*.
\]  

(4)

The size of the largest cluster \( s^* = P_\infty L^d \) is thus proportional to

\[
s^* \sim \text{const} + \log L/|\log \lambda|,
\]  

(5)

in agreement with (2). We can see the behavior predicted by eq. (2) on fig. 3, where we plot the number of sites in the largest cluster \( s^*_\infty \) versus \( \log L \). The change of the slope with \( p \) is due to the dependence of \( \lambda \) on \( p \). Unfortunately, this dependence is only known asymptotically for \( p \to O \) (where \( \lambda \sim p \) \([1]\)) and in the vicinity of \( p_c \) (where it is \( \log \lambda \sim (p_c - p)^{\frac{1}{46}} \)). Eq. (2) implies that the finite size scaling relation (1) is changed by a logarithmic scaling dependence in the case of percolation. This dependence should manifest itself in figs. 1 and 2 as a curvature for large \( e L^{1/\nu} \), which, however, is hard to distinguish within the statistical error. So the validity of the estimates of the exponent \( \gamma \) is limited by the logarithmic correction but the fact that finite size scaling holds is confirmed by our data in figs. 1 and 2.

The difference found in the finite size scaling of the order parameter of the Ising model and of percolation at \( p < p_c \) can be attributed to the different fluctuation behavior of these models. In the Ising model the magnetization scales according to ref. \([2]\) as

\[
M \sim (\langle M^2 \rangle)^{1/2} \sim (\chi_T L^d \langle L^2 \rangle)^{1/2} \sim L^{-d/2},
\]  

(6)

since the susceptibility \( \chi_T \) is finite in the disordered phase. In percolation, by analogy we find

\[
P_\infty \sim (\Delta P_\infty^2)^{1/2} \sim (\chi_p L^d \langle L^2 \rangle)^{1/2},
\]  

(7)

where \( (\Delta P_\infty^2)^{1/2} \) is a fluctuation of order parameter and

\[
\chi_p = L^{-d} \sum_i \langle \gamma_i^w \rangle - \langle \gamma_i^w \rangle^2,
\]  

(8)

with \( \gamma_i^w = 1 \) if the site \( i \) belongs to the infinite cluster and \( \gamma_i^w = O \) otherwise \([10]\). Then by scaling

\[
\chi_p \sim s^w \langle L^d \rangle \sim \log^2 L/L^d,
\]  

(9)

and therefore \( \chi_p \) tends to zero for very large \( L \), unlike the Ising model. This leads to the different behavior of the order parameter \( P_\infty \) (2).

In summary, we found that the finite size scaling of the order parameter of the Ising model and of percolation in the disordered phase are very different. This difference can be attributed to the fact that the fluctuations that one encounters in percolation are not of the same nature as those described in ref. \([2]\) for the Ising model. We see that for \( p < p_c \) in percolation finite size scaling holds, but with a logarithmic dependence that can be seen numerically for large system sizes.

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