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Saturation Transients in Saltation and their Implications on Dunes Shapes [1]

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Abstract – We present a model for dune formation that takes saturation transients in the saltation layer into account. We show that these transients introduce a characteristic length scale that is crucial for dune formation and the macroscopic shape of dunes, albeit determined by intrinsic properties of the saltation kinetics. Finally, we demonstrate that our model can explain the scale dependence of the shape of barchan dunes revealed in recent field measurements.

Résumé – Nous présentons un modèle pour la formation de dunes qui incorpore des transients de saturation dans la couche de saltation et ansi sa dynamique. Nous montrons que ces transients introduisent une longeur characteristique qui est importante pour la formation de dunes, mais qui est basée sur des propriétés dynamiques intrinsèques de la kinétique de saltation. Finalement, nous montrons que notre modèle peut expliquer les différences de forme entre des dunes petites et grandes, ce qu'ont revelé de nouvell mesures.

Keywords – barchan, shape, saltation, saturation, transients

1. Introduction

Sand dunes develop wherever sand is exposed to an agitating medium (air, water . . .) that lifts grains from the ground and entrains them into a surface flow. The diverse conditions of wind and sand supply in different regions on Earth give rise to a large variety of different shapes of aeolian dunes [1, 2, 3]. Moreover, dunes have been found on the sea–bottom and even on Mars. Despite the long history of the subject, the underlying physical mechanisms of dune formation are still not very well understood. How are aerodynamics (hydrodynamics) and the particular properties of granular matter acting together to create dunes? How is the shape of a dune maintained when it moves and maturates? What determines the size of dunes? Due to the fact that neither now nor in the near future we will be able to simulate dunes on the grain scale (an average barchan comprises 10^{15} grains), we concentrate in the following on an effective continuum model that can be applied to sand dunes or other geomorphological problems on a large scale. Due to the highly complicated physical processes involved (saltation, turbulent wind) and the wide range of length and time scales to be covered — from the dynamics of single sand grains, the formation of ripples, to the genensis and migration of dune fields, the time and length scales span over more than seven orders of magnitude — the derivation of such a model

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[1] **Dedicated to Prof. Erwin Müller–Hartmann on the occation of his 60$^{\rm th}$ birthday.**

Figure 1: Barchan dunes near Laâyoune, Morocco. The dune in the front, on the left side was measured in detail during our field trip in May 1999 (no 5).

is far from being a straight–forward task. Strong simplifications are absolutely necessary. In the following we discuss results obtained with a minimal model for aeolian sand dunes to address these questions. By "minimal model" we mean a model that is as simple as possible while retaining the essential ingredients, i.e. which contains those and only those elements necessary for a successful modelling of the phenomenology obtained in field measurements. Although it refers only to rather generic properties of the wind velocity field and the laws of aeolian sand transport, it can make interesting qualitative predictions (that are not sensitive to the simplifying assumptions), e.g. about the surface profile, the development and position of the slipface, dune migration, etc.

2. The shape of barchan dunes

Since Bagnold made his field measurements in the 1930s more than half a century has gone by and many other field data have been acquired. Especially, the barchan dune that is abundant e.g. in the Pampa de La Yoya in southern Peru has been the object of many expeditions [4, 5, 6, 7]. The crescent–like shape of barchan dunes is well known qualitatively. We performed our own measurements of the shape of several barchan dunes in Morocco [8]. The measured dunes have heights between 1.5 m and 10 m, while their bases are typically 40 m to 150 m long and 30 m to 100 m wide. The windward or stoss–side of the dune has typical slopes between 8° and 20° and is limited by a sharp edge, called the brink. The brink coincides in many cases with the crest of the dune and separates the slip face from the dune's windward side. Roughly speaking, its shape as seen from above, is a parabola–like curve reaching from the tip of one horn to the point of maximum slip face height and back to the tip of the other horn, cf. Figure 1. Typical geometric relations of barchan dunes such as height and width or height and length relationship have been measured many times, but the shape has never been seized in its totality. Merely estimating the volume was not possible without further assumptions. In order to obtain precise shape data of barchan dunes we therefore had to performe the mentioned field measurements in Laâyoune, Morocco [8]. In contrast to previously collected data that suggested the shape invariance of barchan dunes, our precise records revealed a difference in

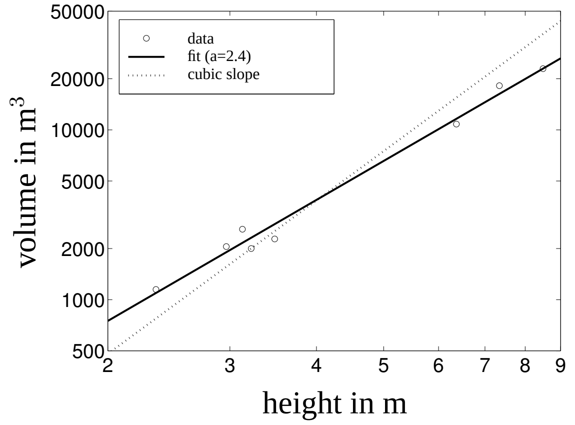

Figure 2: Height and volume relationship. Fitting the function $V = bH^a$ to the data leads to an exponent $\alpha \approx 2.4$ significantly different from the cubic relation expected for scale invariant dunes.

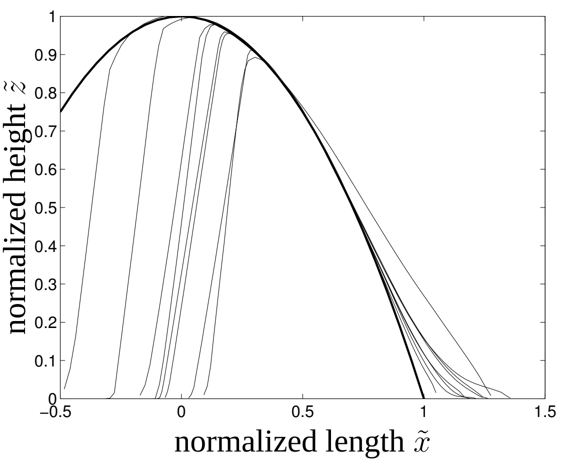

Figure 3: Profile along the symmetry plane of the dunes (thin lines) that have been rescaled to match a parabola (thick line).

shape between small and large barchan dunes. In particular the position of the slip face and the volume of the dunes do not scale trivially as shown in Figure 2 and 3. As a consequence, we can draw the puzzling conclusion that there is a characteristic length scale involved in dune formation. Given the scale invariance of the strongly turbulent wind that creates the dunes, it is *a priori* far from obvious what this scale should be. As we will point out below, to answer this question, it was necessary to understand in considerable detail the underlying sand transport mechanism.

3. The Model

Since we are interested in the formation and movement of dunes, the important time scale of our problem is defined by the erosion processes that changes the surface. A significant change of the surface happens within some hours or even days. In contrast to this, the time scale of the wind and the saltation process is in the order of seconds and therefore several orders of magnitude faster. Hence, we will use in the following stationary solutions for the wind field and the sand flux. Similarly we neglect the finite life time of avalanches (a few seconds) and consider them as instantaneous compared to the movement of the dunes. The separation of the different time scales and the resulting approximations lead to an enormous simplification, because it decouples the different physical processes. The entire model can be thought of as four (almost) independent parts: the stationary wind field over a complex terrain, the stationary aeolian sand transport, the time evolution of the surface due to erosion, and avalanches.

3.1. The wind shear stress

The fully turbulent atmospheric boundary layer develops over a flat surface the well known logarithmic velocity profile $v(z)$ [9],

$$v(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_0}, \tag{1}$$

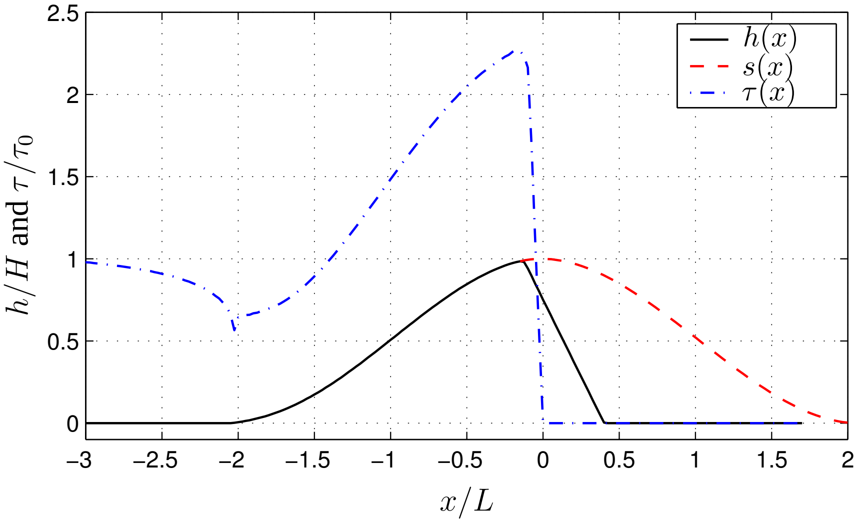

Figure 4: The envelope $\tilde{h}(x)$ of the windward profile of a dune $h(x)$ and the separating streamline $s(x)$ form together a smooth object which is used to calculate the air shear stress $\tau(x)$ on the windward side. In the region of recirculation the air shear stress τ is set to zero.

where u_* denotes the shear velocity, z_0 the roughness length of the surface, and $\kappa = 0.4$ the von Kármán constant. The shear velocity u_* has the dimensions of a velocity but is defined by the shear stress $\tau = \rho_{\text{air}} u_*^2$ and the density ρ_{air} of the air. A perturbation of the ground $h(x)$ such as a dune or hill gives rise to a non–local perturbation $\hat{\tau}(x)$ of the undisturbed air shear stress τ_0,

$$\tau(x) = \tau_0 \left[1 + \hat{\tau}(x)\right]. \tag{2}$$

The functional dependence of the air shear stress perturbation is crucial for the theoretical understanding of the stability of dunes and to predict the sand flux onto the windward side of a dune. Analytical calculations of the flow over a gentle hill yield an analytical expression for the shear stress perturbation $\hat{\tau}(x)$ [10, 11, 12]. We performed further simplifications in order to obtain a minimal expression that captures the crucial features (and only those) and is applicable for sand dunes [13]. The resulting formula for the air shear stress perturbation $\hat{\tau}$ is,

$$\hat{\tau}(x) = A\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h'}{x-\xi}d\xi + Bh'\right), \tag{3}$$

where h' denotes the spatial derivative of the dune's profile $h(x)$ in wind direction. The coefficients $A(L/z_0)$ and $B(L/z_0)$ depend only logarithmically on the ratio between the characteristic length L of the dune and the roughness length z_0 of the surface. For a dune with a length and width ratio $W/L \approx 1$ and $L/z_0 = 4.0\,10^5$ we obtain $A \approx 3.2$ and $B \approx 0.3$ from Ref. [13, 14]. Equation (3) has several features that are important for dune formation. First, the air shear stress is completely scale invariant and leads to the same speed–up for small and large dunes. This is expected in the fully turbulent regime where no characteristic length exists. Secondly, the shear stress perturbation $\hat{\tau}(x)$, Equation (3), scales with the height H and inversely with the characteristic length L of the dune and thus with the average slope of the dune's windward side, $\hat{\tau} \propto H/L$. Thirdly, a depression of $\tau(x)$ in front of the hill occurs as a consequence of

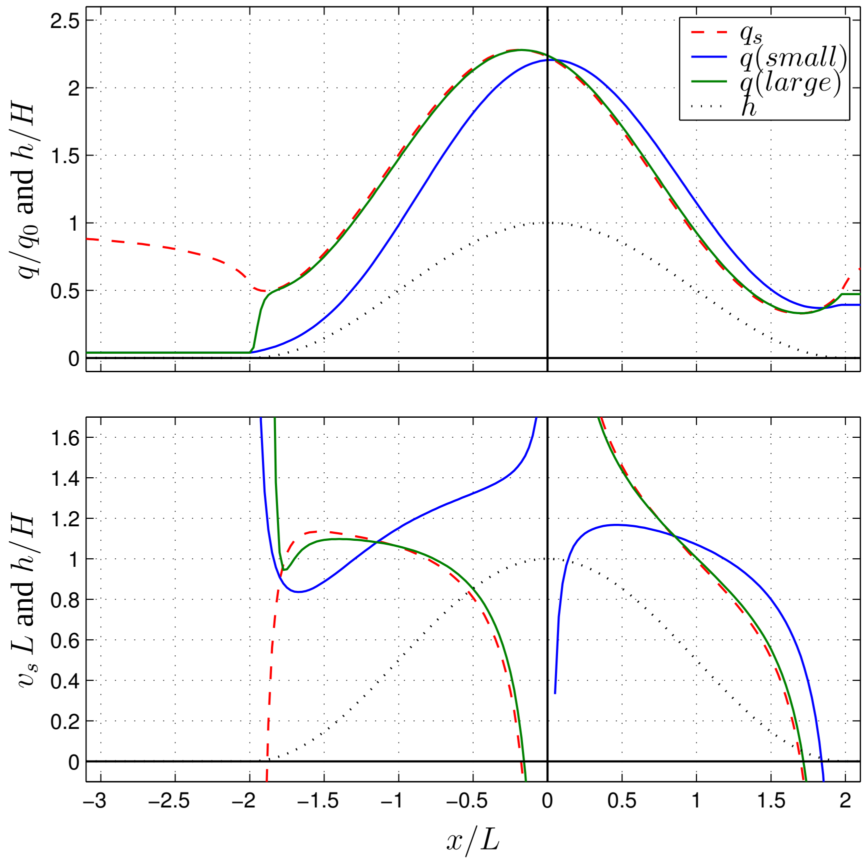

Figure 5: Top: saturated sand flux $q_s(x)$ and the flux $q(x)$ according to Equation (5), including saturation transients, for two cosine shaped hills with $H/L = 1/8$ and height of $H = 1\,m$ and $H = 10\,m$, respectively. The saturation length is $l_s = 0.8\,m$. The saturated flux q_s is scale invariant, whereas the actual flux q differs is scale dependent. Bottom: The corresponding surface velocities $v_s(x)$, which indicate that the small heap flattens ($\partial_x v_s > 0$) and the large heap steepens ($\partial_x v_s < 0$).

the strongly non–local contribution in Equation (3). Finally, the most important feature is the symmetry breaking contribution, Bh'. The symmetry breaking leads to an upwind shift of the maximum of the air shear stress with respect to the maximum of the hill. The physical reason for the broken symmetry is the inertia of the air represented by the non–linear convection term in the stationary Navier Stokes equation. The upwind shift of the air shear stress is indispensable for the growth of dunes, where deposition occures at the crest [14].

Equation (3) is based on a perturbation theory and can only be applied to smooth hills. Jackson and Hunt [10] assumed $H/L < 0.05$, whereas Carruthers [15] showed that mean slopes up to $H/L \approx 0.3$ give reasonable results. The windward side of a barchan dune is always below the latter value and the formula should be applicable. However, flow separation occurs at the brink, which is out of the scope of the linear perturbation theory. A heuristic solution to solve this problem has been suggested by Zeman and Jensen [16]. They introduced a separation bubble that comprises the recirculating flow (the large eddy in the wake of the dune), which reaches from the brink (the point of detachment) to the bottom (to the point of reattachment). We model the separating streamline by a third order polynomial that is a smooth continuation of the profile $h(x)$ at the brink x_{brink} and at the reattachment point $x_{brink}+L_r$, i.e. $h(x_{brink}) = s(0)$,

$h'(x_{brink}) = s'(0)$, $s(L_r) = 0$, $s'(L_r) = 0$, where $L_r \approx 6H$ is the downwind distance of the reattachment point from the brink. The shear stress perturbation $\hat{\tau}(x)$ for the windward side of the dune is finally calculated using Equation (3), the profile $h(x)$ on the windward side, and the separating streamline $s(x)$ on the lee side. An example is depicted in Figure 3.1.

3.2. The Sand flux

Sand transport has been studied already by Bagnold [17] and it was also him who proposed the first phenomenological law that predicted the sand transport from the shear stress of the air. Improved laws have been proposed by several authors in the meantime [18, 19, 20]. However, all these relations assume that the sand flux q is in equilibrium and can be written as a function of the shear stress τ, $q\left(\tau(x)\right)$. Temporal or spatial transients are completely neglected. In the following we will call such a relation *saturated*, because it predicts the amount of sand that can be maintained in the saltaion layer at a certain air shear stress τ, i.e. if the external conditions change and the sand flux was under– or over–saturated, erosion or deposition would occur until the new saturated value is reached. Many different functional forms of these sand transport laws exist and have been used in the past. However, for high shear stresses they all converge to the simple relation proposed by [1],

$$q \propto \tau^{3/2}. \tag{4}$$

All other more elaborate relations add higher order corrections to the Bagnold formula that become important close to the air shear stress threshold. To overcome the limitation of saturation and to obtain information about the dynamics of the saltation process, numerical simulations on the grain scale have been performed in the last years [21, 22, 23]. Still, concerning the modeling of dune formation, both approaches had to be discarded. The microscopic models are computationally too expensive and the equilibrium assumption that is inherent in the simple flux relations does not hold on the entire windward side of a dune [24, 12, 25, 26, 27]. Since both known approaches cannot be used to model dune formation we developed a new phenomenological continuum saltation model that is computationally very efficient on the one side and on the other side incorporates the dynamics of the saltation layer and thus allows for saturation transients [26]. In this model the sand flux is defined by a differential equation of the form

$$\frac{\partial}{\partial x} q = \frac{1}{l_s} q \left(1 - \frac{q}{q_s}\right), \tag{5}$$

where $q_s(\tau)$ is the saturated sand flux and $l_s(\tau)$ the characteristic length of the saturation transients, called saturation length. The saturation length $l_s(\tau)$ depends on the air shear stress, but converges towards a constant value for $\tau \gg \tau_t$ [26]. A comparison between the saturated sand flux and our model, Equation (5), calculated on cosine shaped hills of different size, can be seen in Figure 5 (top).

A spatial change in the sand flux causes erosion or deposition and leads to a change in the shape and/or a displacement of the dune. To analyze this quantitatively, we define the local surface velocity $v_s(x)$ according to mass conservation and convection,

$$v_s(x) \frac{\partial h}{\partial x} = \frac{1}{\rho_{\text{sand}}} \frac{\partial q}{\partial x}. \tag{6}$$

When a dune migrates shape invariantly the surface velocity $v_s(x)$ is constant. An increasing surface velocity ($\partial v_s/\partial x > 0$) flattens an object whereas a decreasing surface velocity ($\partial v_s/\partial x < 0$) steepens it. Hence, we can already predict qualitatively whether the surface steepens, flattens, or does keep its shape from a single stationary solution of the surface velocity $v_s(x)$. For instance, from the solutions depicted in Figure 5 (bottom) we can predict that a small shape invariantly moving dune will have a lower height length aspect ratio H/L than a large one.

3.3. The surface evolution

A spatial change in sand flux implies that erosion or deposition takes place and the surface changes in height. The time evolution of the surface can be calculated from the conservation of mass,

$$\frac{\partial h}{\partial t} = \frac{1}{\rho_{\text{sand}}} \frac{\partial q}{\partial x}, \tag{7}$$

where ρ_{sand} is the bulk density of dune sand. Finally, we note that Equation (7) is the only remaining time dependent equation and thus defines the time scale of the model.

The full dune model can be sketched as follows. An initial surface h is used to start the time evolution. If flow separation has to be modeled the separating streamline $s(x)$ is calculated. Next, the air shear stress $\tau(x)$ onto the given surface h (or h and s) is calculated using Equation (2). From the air shear stress $\tau(x)$ the sand flux can be determined using Equation (5). Then, the integration forward in time of the surface is calculated from the mass conservation, Equation (7). Finally, sand is eroded and transported downhill if the local angle $\partial_x h$ exceeds the angle of repose [14]. This redistribution of mass (avalanches) is performed until the surface slope has relaxed below the critical angle. The time integration is calculated until the final shape invariantly moving solution is obtained.

4. Results

In order to analyze the properties of the shape invariantly moving solution of our model, we performed a series of calculations varying the volumes of the Gaussian hills that have been used as initial configuration. The final shape invariantly moving solutions are displayed in Figure 6. For small volumes we obtained heaps without a slip face, whereas for large volumes dunes with a slip face developed. Hence, there is a minimal height for dune formation or, more precisely, a minimal height for the formation of a slip face. Empirically, this was observed many times in nature. The height of the heaps increases over–proportionally with their volume, which leads to a steepening with increasing volume. In order to understand this result we need to analyze the scaling of the air shear stress $\tau(x)$, Equation (3), and the sand flux $q(x)$, Equation (5). Fortunately, the problems of flow separation and avalanches do not occur in the case of heaps. The air shear stress $\tau(x)$ is shifted upwind with respect to the profile $h(x)$ of the dune as seen above. The sand flux $q(x)$ lags a certain distance behind the shear stress of the air. In the steady state, these two shifts must compensate. The lag of the sand flux is an intrinsic property of the saltation process and therefore independent of the shape of the hill. More precisely, it is proportional to the saturation length l_s which is approximately constant for shear stresses not too close to the threshold. In contrast, the shear stress of the air and thus the upwind shift scale with the characteristic length L of the hill. Therefore, the hill has to steepen until the upwind shift of the shear stress matches the lag of the sand flux (for a given influx and wind velocity).

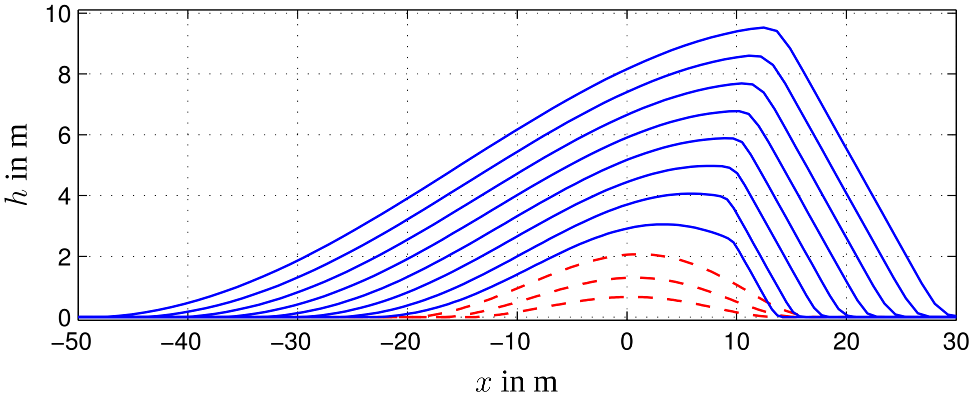

Figure 6: The solutions for large volumes — above a critical heigth — are dunes including a slip face, whereas for small volumes heaps develop. An important fact is that the steepest lee side of a heap (dashed lines) is approximately 15°, which is well below the angle of repose of 34°.

Hence, the hill grows over–proportionally in height and thus steepens with increasing volume. Finally, the slope reaches a critical value and a slip face develops.

The shape of the steady state solutions of dunes depticted in Figure 6 is obviously not scale invariant, but seems to converge to an asymptotic shape for large dunes ($L \gg l_s$). The small dunes just above the critical size are whale shaped, whereas the crest and brink coincides for large dunes. This is in accord with our measurements in Morocco [8].

We want to emphasize that in the steady state the slip face always has a finite size. The reason for this is the separation bubble that is attached to the dune when a slip face exists. The shear stress is calculated using the envelope of dune and separation bubble. Hence, a dune with a slip face has an effective volume that is increased by the volume of the separation bubble.

5. Conclusion

We have shown that saturation transients in the saltation layer have crucial implications on the entire physics of sand dunes. Saturation transients are an intrinsic dynamic property of the microscopic saltation process, yet they break the scale invariance of the entire dune model and introduce a characteristic length scale that we called “saturation length”. This saturation length is different from the saltation length, but related to it, and has a complex, dependence on the air shear stress [26, 14]. By solving our model numerically we have demonstrated that a direct consequence of the saturation length is the shape difference between small and large dunes. This agrees with our field measurements performed in Morocco [8]. The factors that establish the minimal height of a barchan dune have also been a longtime outstanding question. We could show that a competition of the saturation length (the distance that the sand lags behind) and the symmetry breaking of the air shear stress is responsible this minimal height. Hence, the minimal height of a barchan dune is directly related to the dynamics of saltation and thus to microscopic quantities such as the grain trajectories and grain bed collisions. For details we refer the reader to Refs. [13, 14].

A general feature of a continuum model such as the sand flux model that has been discussed here is that phenomenological parameters have to be determined. More precisely, in our continuum saltation model the microscopic information such as the interaction between grain/bed and grain/air are incorporated in two phenomenological functions: the equilibrium flux $q_s(\tau)$ and the saturation length $l_s(\tau)$ which both depend on the air shear stress. We obtained them by deriving their approximate functional structure and determining the free parameters according to wind tunnel data [26, 14]. However, if such data are not available, e.g. for dunes on mars, these parameters (functions) are completely unknown. Microscopic simulation on the grains scale, e.g. molecular dynamics coupled with fluid dynamics, could provide the missing information. Such a simulation would only depend on microscopic quantities, e.g. the grain diameter and the density of the fluid/grains, and could predict the phenomenological functions $q_s(\tau)$ and $l_s(\tau)$ for the macroscopic model without further assumptions.

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